

AMENABILITY OF GROUPOIDS ARISING FROM PARTIAL SEMIGROUP ACTIONS AND TOPOLOGICAL HIGHER RANK GRAPHS

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ABSTRACT. We consider the amenability of groupoids G equipped with a group valued cocycle $c : G \rightarrow Q$ with amenable kernel $c^{-1}(e)$. We prove a general result which implies, in particular, that G is amenable whenever Q is amenable and if there is countable set $D \subset G$ such that $c(G^u)D = Q$ for all $u \in G^{(0)}$.

We show that our result is applicable to groupoids arising from partial semigroup actions. We explore these actions in detail and show that these groupoids include those arising from directed graphs, higher rank graphs and even topological higher rank graphs. We believe our methods yield a nice alternative groupoid approach to these important constructions.

1. INTRODUCTION

It is often important to establish the amenability of groupoids that arise in applications. For example, amenability implies the equality of the reduced and universal norms on the associated groupoid algebras. This is important in the study of the Baum-Connes conjecture for groupoid C^* -algebras as it is the reduced algebra that plays the key role, while the universal algebra has the better functorial properties. For example, Tu has shown that the C^* -algebra of an amenable groupoid with a Haar system satisfies the Baum-Connes conjecture and the UCT [24]. In the classification program, amenability implies nuclearity which is typically a crucial hypothesis.

The sort of groupoids we wish to focus on are those arising from the much studied C^* -algebras associated to higher-rank graphs, and more recently, to topological higher-rank graphs. As a specific example, we recently considered the C^* -algebras of groupoids associated to k -graphs (see [20]). Such groupoids have a canonical \mathbf{Z}^k -valued cocycle c , and it is not hard to see that $c^{-1}(0)$ is amenable. In some cases, c is not only surjective, but strongly surjective in that $c(G^u) = \mathbf{Z}^k$ for all $u \in G^{(0)}$. Then the amenability of G is a consequence of [1, Theorem 5.3.14]. However, there are interesting examples in which c need not be strongly surjective, and examples show that the problem of the amenability of G turns out to be very subtle. A result of Spielberg [22, Proposition 9.3] asserts that if G is étale and if $c : G \rightarrow Q$ is a continuous cocycle into a countable amenable group, then G is amenable whenever $c^{-1}(e_Q)$ is. Although this result is satisfactory for most k -graphs, the proof is unsatisfying in that it circumvents groupoid theory by invoking

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the nontrivial result that for étale groupoid, $C^*(G)$ is nuclear if and only if G is amenable [15, Corollary 2.17]. In particular, this result is valid only for étale groupoids.

Here we want to prove a general result that subsumes both cocycle results from [1] and from [22]. That such a result will have delicate hypotheses is foreshadowed by the observation that one can have a surjective continuous cocycle c from a groupoid into an amenable group Q such that $c^{-1}(e_G)$ amenable and still have G fail to be amenable. For example, let Q be an amenable group with a nonamenable subgroup N (which obviously is not closed in Q). Let G be the group bundle $G = Q \amalg N$ and let c be the identity map: $c(\gamma) = \gamma$.

Nevertheless, we obtain a quite general result: Theorem 4.2. It implies in particular, that if $c : G \rightarrow Q$ is a continuous cocycle into an amenable group Q with amenable kernel such that there is a countable set D so that $c(G^u)D = Q$ for all $u \in G^{(0)}$, then G is amenable. Even in this form, we recover both results above and remove the hypothesis that G be étale from Spielberg's result.

Although our results apply to topological groupoids with Haar systems, it is interesting that our proof relies on the notions of Borel groupoids, Borel amenability and Borel equivalence. At a crucial juncture, we are able to show that our groupoid is Borel equivalent to a Borel amenable groupoid. Since we also show that Borel equivalence preserves Borel amenability, we can appeal to the result from [19] which demonstrates that topological amenability is equivalent to Borel amenability — at least for locally compact groupoids with Haar systems.

Having proved our cocycle result, it is crucial to show how it can be applied to groupoids which are currently being studied in the literature. To do this, we show that our results can be applied to groupoids arising from (partial) semigroup actions. A key rôle is played by the notion of *directed* action (Definition 5.2) which gives a partition of the space into orbits. We explore these actions in detail. Then, making significant use of hard work due to Nica and Yeend, we are able to show that such groupoids include those arising from directed graphs, higher rank graphs and even topological higher rank graphs. We think our methods using semigroup actions yield a nice alternative groupoid approach to these important constructions.

To prove our cocycle result, we work with locally Hausdorff, locally compact groupoids which are always assumed to be second countable and to possess a Haar system. Note that a second countable, locally Hausdorff, locally compact groupoid is the countable union of compact Hausdorff sets. Hence its underlying Borel structure is standard. In Section 2 we review the notion of Borel amenability and some of the basic properties of Borel groupoids we need in the sequel. In Section 3 we recall the notion of Borel equivalence and prove that it preserves Borel amenability. In Section 4, we prove the main amenability result. In Section 5 we introduce semigroup actions. When the action is directed, there is a semi-direct product groupoid. This condition puts into light two classes of semigroups: Ore semigroups and quasi-lattice ordered semigroups. Under reasonable hypotheses, our main result applies and the semi-direct product groupoid is amenable. In Section 6, we show how the groupoid corresponding to a topological higher rank graph (and therefore to the many subcases of topological higher rank graphs) can be realized as the semi-direct product groupoid of a directed semigroup action of $P = \mathbb{N}^d$. In fact, we consider P -graphs, where P is an arbitrary semigroup rather than simply \mathbb{N}^d as in the original definition. The natural assumption is that the semigroup be quasi-lattice ordered.

If moreover the semigroup is a subsemigroup of an amenable group and the graph satisfies a properness condition, then this groupoid is amenable as well. Besides the amenability problem, the section reveals a tight connection between higher rank C^* -algebras and Wiener-Hopf C^* -algebras. In fact, we use the techniques introduced by Nica in [14]; in particular the Wiener-Hopf groupoid of a quasi-lattice ordered semigroup defined by Nica appears as a particular case of our general construction for topological higher rank graphs.

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2. BOREL AMENABILITY

For the details on Borel groupoids, proper Borel amenability and proper Borel G -spaces, we refer to [1] and to [1, §2.1a] in particular. Recall that in order for a groupoid to act on (the left) a space X , we require a map $r_X : X \rightarrow G^{(0)}$. If there is no ambiguity, we write simply r in place of r_X and call it the *moment map*.¹ As in [1], to avoid pathologies we will always assume that our Borel spaces are analytic Borel spaces. We recall some of the basics here. If X and Y are Borel spaces and $\pi : X \rightarrow Y$ is Borel surjection, then a π -system is a family of measures $m = \{m^y\}_{y \in Y}$ such that each m^y is supported in $\pi^{-1}(y)$ and such that

$$y \mapsto \int_X f(x) dm^y(x)$$

is Borel for any nonnegative Borel function f on X . If G is a Borel groupoid, X and Y are Borel G -spaces and π is G -invariant, then we have that m is *invariant* if $\gamma \cdot m^y = m^{\gamma \cdot y}$ whenever $s(\gamma) = r(y)$. (By definition, $\gamma \cdot m^y(E) = m^y(\gamma^{-1} \cdot E)$.)

Definition 2.1 ([1, Definition 2.1.1(b)]). Suppose that G is a Borel groupoid and that X and Y are Borel G -spaces. A surjective Borel G -map $\pi : X \rightarrow Y$ is *G -properly amenable* if there is a invariant Borel π -system $m = \{m^y\}_{y \in Y}$ of probability measures on X . Then we say that m is an *invariant mean for π* .

Remark 2.2. Notice that if $\pi : X \rightarrow Y$ is a Borel G -map, then the induced map $\dot{\pi} : G \backslash X \rightarrow G \backslash Y$ is Borel with respect to the quotient Borel structures. Suppose that π is G -properly amenable and that $\{\lambda^y\}$ is an invariant mean for π . Let $p : X \rightarrow G \backslash X$ be the quotient map. Then the push forward $p_* \lambda^y$ is a probability measure supported on $\dot{\pi}^{-1}(\dot{y})$. By invariance, it depends only on \dot{y} and we can denote this measure by $\dot{\lambda}^{\dot{y}}$. If b is a bounded Borel function on $G \backslash Y$, then

$$\int_{G \backslash Y} b(\dot{z}) d\dot{\lambda}^{\dot{y}}(\dot{z}) = \int_X b(p(x)) d\lambda^y(x).$$

Hence $\{\dot{\lambda}^{\dot{y}}\}_{\dot{y} \in G \backslash Y}$ is a Borel $\dot{\pi}$ -system of probability measures on $G \backslash X$.

Definition 2.3 ([1, Definition 2.1.2]). A Borel groupoid G is *proper* if the range map $r : G \rightarrow G^{(0)}$ is G -properly amenable. A Borel G -space X is *proper* if the projection $p : X * G \rightarrow X$ is G -properly amenable.

¹The term projection map, anchor map and structure map have also been used in the literature.

Remark 2.4. Thus X is a proper G -space if and only if there is a family $\{m^x\}_{x \in X}$ of probability measures m^x on G supported in $G^{r(x)}$ such that $x \mapsto m^x(f)$ is Borel for all non-negative Borel functions on G and such that $\gamma \cdot m^x = m^{\gamma \cdot x}$.

If X and Y are (left) G -spaces, then the fibered product $X * Y$ is a G -space with respect to the diagonal action. If X and Y are Borel G -spaces, then we give $X * Y$ the Borel structure as a subset of $X \times Y$ with Borel structure generated by the Borel rectangles. In particular, if X is a G -space, then $X * G = \{(x, \gamma) \in X \times G : r(x) = r(\gamma)\}$ is a G -space with respect to the diagonal action.²

Definition 2.5. Suppose that G is a Borel groupoid and that X and Y are Borel G -spaces. A surjective Borel G -map $\pi : X \rightarrow Y$ is *G -amenable* (or simply *amenable*) if there is no ambiguity about G) if there is a sequence $\{m_n\}_{n \in \mathbf{N}}$ of Borel systems of probability measures $y \mapsto m_n^y$ on X which is approximately invariant in the sense that for all $\gamma \in G$, $\|\gamma \cdot m_n^y - m_n^{\gamma \cdot y}\|_1$ converges to 0, where $\|\cdot\|_1$ is the total variation norm. We say that $\{m_n\}_{n \in \mathbf{N}}$ is an *approximate invariant mean* for π .

The notion of Borel amenability for groupoids was first formalized in [19, Definition 2.1] — however, here we use the formulation in which conditions (ii) and (iii) of [19, Definition 2.1] have been replaced by (ii').

Definition 2.6. A Borel groupoid G is (Borel) *amenable* if the range map $r : G \rightarrow G^{(0)}$ is G -amenable. A Borel G -space X is *amenable* if the projection $p : X * G \rightarrow X$ is G -amenable.

A key result about Borel amenable maps we need is the following.

Lemma 2.7. *Let G be a Borel groupoid. If there is a proper Borel G -space X such that the moment map $r : X \rightarrow G^{(0)}$ is Borel amenable, then G is Borel amenable.*

Proof. Since, by assumption, the map $\pi_X : X * G \rightarrow X$ is properly amenable, there is an invariant system $m = \{m^x\}_{x \in X}$ of probability measures for π_X . We can view each m^x as a measure on $G^{r(x)}$ such that $\gamma \cdot m^x = m^{\gamma \cdot x}$. Since r_X is amenable, there is an approximate invariant mean $\{\mu_n\}$. Then μ_n^u is a probability measure on $r_X^{-1}(u)$, and for all $\gamma \in G$, $\lim_n \|\gamma \cdot \mu_n^{s(\gamma)} - \mu_n^{r(\gamma)}\|_1 = 0$. Define

$$\lambda_n^u = \int_X m^x d\mu_n^u(x).$$

Then λ_n^u is a probability measure supported in G^u . The system $\lambda_n = \{\lambda_n^u\}_{u \in G^{(0)}}$ is certainly Borel, and

$$\begin{aligned} \|\gamma \cdot \lambda_n^{s(\gamma)} - \lambda_n^{r(\gamma)}\|_1 &= \left\| \int_X \gamma \cdot m^y d\mu_n^{s(\gamma)}(y) - \int_X m^x \mu_n^{r(\gamma)}(x) \right\| \\ &= \left\| \int_X m^{\gamma \cdot x} d\mu_n^{s(\gamma)}(y) - \int_X m^x d\mu_n^{r(\gamma)}(x) \right\| \\ &= \left\| \int_X m^x d(\gamma \cdot \mu_n^{s(\gamma)} - \mu_n^{r(\gamma)})(x) \right\| \\ &\leq \|\gamma \cdot \mu_n^{s(\gamma)} - \mu_n^{r(\gamma)}\|_1. \end{aligned}$$

²In fact, $X * G$ is a groupoid — sometimes called the transformation groupoid for G acting on X . This groupoid is denoted by $X \rtimes G$ in [1]. After identifying its unit space with X , the range map $(x, \gamma) \mapsto x$ is clearly G -equivariant. Hence, in Definition 2.3, we defined a G -space X to be properly amenable exactly when the groupoid $X \rtimes G$ is properly amenable.

It follows that λ is an approximate invariant mean for $r : G \rightarrow G^{(0)}$. Thus G is Borel amenable. \square

Of course, a proper Borel groupoid is Borel amenable. Our next lemma should be compared with [1, Proposition 5.3.37].

Lemma 2.8. *Suppose that G is a Borel groupoid which is the increasing union of a sequence of proper Borel subgroupoids G_n . Then G is Borel amenable.*

Proof. By assumption, we can find a system $m_m = (m_n^u)_{u \in G_n^{(0)}}$ of invariant probability measures on G_n ; that is, $\gamma \cdot m_n^{s(\gamma)} = m_n^{r(\gamma)}$ for all $\gamma \in G_n$. Using the standard theory of disintegration of measures for example (see [25, Theorem I.5]), we can extend each m_m to a Borel system of probability measures on all of $G^{(0)}$. Then that $m = (m_n)$ is an approximate invariant mean for $r : G \rightarrow G^{(0)}$. \square

3. BOREL EQUIVALENCE

The definition of equivalence for Borel groupoids is given in the appendix of [1]. In light of [19], it turns out to be a much more significant notion than originally thought.

Definition 3.1 ([1, Definition A.1.11]). Let G and H be Borel groupoids. A (G, H) -Borel equivalence is a Borel space Z such that

- (a) Z is a free and proper left Borel G -space,
- (b) Z is a free and proper right Borel H -space,
- (c) The G - and H -actions commute,
- (d) $r : Z \rightarrow G^{(0)}$ induces a Borel isomorphism between Z/H and $G^{(0)}$, and
- (e) $s : Z \rightarrow H^{(0)}$ induces a Borel isomorphism between $G \backslash Z$ and $H^{(0)}$.

In this case, we say that G and H are *Borel equivalent*.

It is proved in [1, Theorem 2.2.17] that equivalence of locally compact groupoids preserves (topological) amenability. Here we show a similar result holds in the Borel case using virtually the same argument.

Theorem 3.2. *Suppose that G and H are equivalent Borel groupoids. If H is Borel amenable (resp., properly amenable), then so is G .*

Proof. Let Z be a (G, H) -equivalence. Consider the commutative diagram

$$\begin{array}{ccc}
 G & \xrightarrow{r} & G^{(0)} \\
 p_G \uparrow & & \uparrow r \\
 Z * G & \xrightarrow{p_Z} & Z \\
 p \downarrow & & \downarrow q \\
 Z & \xrightarrow{q} & G \backslash Z,
 \end{array}$$

where $p(z, \gamma) := \gamma^{-1} \cdot z$.

First, assume that G is properly amenable. To see that H is properly amenable it will suffice, by [1, Corollary 2.1.7], to see that that H -map $s : Z \rightarrow H^{(0)}$ is properly amenable.

Let $\lambda = \{\lambda^u\}_{u \in G^{(0)}}$ be an invariant mean for $r : G \rightarrow G^{(0)}$. We first build an invariant mean λ_Z for $p_Z : Z * G \rightarrow Z$: let λ_Z^z be given by

$$\lambda_Z^z(f) = \int_G f(z, \lambda) d\lambda^{r_Z(z)}(\gamma).$$

Consider the measure on Z obtained by the push forward $q_*\lambda_Z^z$. Since invariance implies $\eta^{-1} \cdot \lambda^{r(\eta)} = \lambda^{s(\eta)}$ for all $\eta \in G$, it follows that

$$\begin{aligned} q_*\lambda_Z^{\eta \cdot z}(b) &= \int_G b(p(\eta \cdot z, \gamma)) d\lambda_Z^{\eta \cdot z}(\gamma) \\ &= \int_G b(\gamma^{-1}\eta \cdot z) d\lambda^{r_Z(\eta \cdot z)}(\gamma) = \int_G b(\gamma^{-1} \cdot z) d(\eta^{-1} \cdot \lambda^{r(\eta)})(\gamma) \\ &= q_*\lambda_Z^z(b). \end{aligned}$$

Hence the push forward $q_*\lambda_Z^z$ depends only on z in $G \backslash Z$. Since Z is a Borel equivalence, we can identify the quotient map $q : Z \rightarrow G \backslash Z$ with the moment map $s : Z \rightarrow H^{(0)}$. Thus we get an s -system of measures $\nu = \{\nu^v\}_{v \in H^{(0)}}$ where $\nu^v = q_*\lambda_Z^z$ for any z with $s(z) = v$. It will suffice to see that ν is invariant. But if $s(z) = r(h)$, then

$$\begin{aligned} \int_Z b(z) d(\nu^{r(h)} \cdot h)(z) &= \int_G b((\gamma^{-1} \cdot z) \cdot h) d\lambda^{r(z)}(\gamma) = \int_G b(\gamma^{-1} \cdot (z \cdot h)) d\lambda^{r(z \cdot h)}(\gamma) \\ &= \int_Z b(z) d\nu^{s(h)}(z). \end{aligned}$$

This completes the proof for proper amenability.

Now we assume that G is Borel amenable and that $\lambda = \{\lambda_n\}$ is an approximately invariant mean for $r : G \rightarrow G^{(0)}$ with $\lambda_n = \{\lambda_n^u\}_{u \in G^{(0)}}$. By Lemma 2.7, it will suffice to show that the H -map $s : Z \rightarrow H^{(0)}$ is Borel amenable. The argument parallels that above argument for proper amenability.

We begin by lifting each λ_n to a system $\lambda_{n,Z} = \{\lambda_{n,Z}^z\}_{z \in Z}$ of probability measures for $p_Z : Z * G \rightarrow Z$:

$$\lambda_{n,Z}^z(f) = \int_G f(z, \gamma) d\lambda_n^{r(z)}(\gamma).$$

We let $\mu_n^z = q_*\lambda_{n,Z}^z$. Since $\lambda_{n,Z}$ is not necessarily invariant, we can't assert that μ_n^z depends only on z . However

$$\int_Z b(w) d\mu_n^z(w) = \int_G b(\gamma^{-1} \cdot z) d\lambda_n^{r(z)}(\gamma),$$

and μ_n^z is supported on $q^{-1}(z) = G \cdot z$.

Since Z is a proper G -space, by definition there is a G -invariant system of probability measures $\rho = \{\rho^z\}_{z \in Z}$ for the Borel G -map $p_Z : Z * G \rightarrow Z$ which we view as a family of measures on G such that $\text{supp } \rho^z \subset G^{r(z)}$. As in Remark 2.2, ρ drops to a system $\dot{\rho} = \{\dot{\rho}^z\}_{z \in G \backslash Z}$ for the quotient map $q : Z \rightarrow G \backslash Z$. Since $(z, \gamma) \mapsto \gamma^{-1} \cdot z$ identifies $G \backslash (Z * G)$ with Z , we can view these as measures on Z . Explicitly, $\dot{\rho}^z = p_*\rho^z$ and

$$\dot{\rho}^z(b) = \int_G b(\gamma^{-1} \cdot z) d\rho^z(\gamma).$$

Using the invariance of ρ we get measures depending only on \dot{z} supported on $q^{-1}(\dot{z})$ by averaging the μ_n^z with respect to $\dot{\rho}^{\dot{z}}$:

$$(3.1) \quad \nu_n^{\dot{z}} = \int_Z \mu_n^z d\dot{\rho}^{\dot{z}}(z) = \int_G \mu_n^{\gamma^{-1} \cdot z} d\rho^z(\gamma).$$

As in the first part of the proof, we use the fact that Z implements an equivalence to identify the quotient map $q : Z \rightarrow G \backslash Z$ with the moment map $s : Z \rightarrow H^{(0)}$. Thus we get an s -system $\{\nu^v\}_{v \in H^{(0)}}$ where $\nu^v = \nu^{\dot{z}}$ for any z such that $s(z) = v$. We will complete the proof by showing that ν is an approximately invariant mean for s . Therefore we need to see that for all $h \in H$, $\|\nu_n^{r(h)} \cdot h - \nu_n^{s(h)}\|_1$ tends to zero with n . Notice that if $(z, \gamma) \in Z * G$, then

$$\|\mu_n^z - \mu_n^{\gamma^{-1} \cdot z}\|_1 \leq \|\lambda_n^{r(\gamma)} - \gamma \cdot \lambda_n^{s(\gamma)}\|_1.$$

Next we claim that for fixed $z \in Z$, $\lim_n \|\nu^{\dot{z}} - \mu_n^z\|_1 = 0$. To see this we employ (3.1), view ρ^z as a measure on $G^{r(z)}$, and deduce that

$$(3.2) \quad \|\nu_n^{\dot{z}} - \mu_n^z\|_1 \leq \int_G \|\mu_n^{\gamma^{-1} \cdot z} - \mu_n^z\| d\rho^z(\gamma) \leq \int_G \|\lambda_n^{r(\gamma)} - \gamma \cdot \lambda_n^{s(\gamma)}\|_1 d\rho^z(\gamma).$$

Since for each γ , $\lim_n \|\lambda_n^{r(\gamma)} - \gamma \cdot \lambda_n^{s(\gamma)}\|_1 = 0$, we see that (3.2) goes to zero by the Lebesgue Dominated Convergence Theorem.

Now fix $h \in H$ and $\epsilon > 0$. Let $z \in Z$ be such that $s(z) = r(h)$ and let $z' = z \cdot h$ and observe that $\mu_n^z \cdot h = \mu_n^{z'}$. Let M be such that $n \geq M$ implies that

$$\|\nu_n^{\dot{z}'} - \mu_n^{z'}\| < \frac{\epsilon}{2}.$$

Then for $n \geq M$, we have

$$\begin{aligned} \|\nu_n^{r(h)} \cdot h - \nu_n^{s(h)}\|_1 &\leq \|\nu_n^{r(h)} \cdot h - \mu_n^z \cdot h\|_1 + \|\mu_n^z \cdot h - \mu_n^{\dot{z}}\| + \|\mu_n^{\dot{z}} - \nu_n^{s(h)}\| \\ &= \|\nu_n^{z'} - \mu_n^{z'}\|_1 + 0 + \|\mu_n^{z'} - \nu_n^{z'}\| < \epsilon. \end{aligned}$$

Thus $s : Z \rightarrow H^{(0)}$ is Borel amenable. This completes the proof. \square

We close this section with two technical results which will be of use in the next section. Recall from [19, Definition 2.3] that a *Borel approximate invariant density* on G is a sequence (g_n) of non-negative Borel functions on G such that

$$\begin{aligned} \int_G g_n(\gamma) d\lambda^u(\gamma) &\leq 1 \quad \text{for all } n, \quad \int_G g_n(\gamma) d\lambda^u(\gamma) \rightarrow 1 \quad \text{for all } u \in G^{(0)} \text{ and} \\ \int_G |g_n(\gamma^{-1}\gamma') - g_n(\gamma')| d\lambda^{r(\gamma)}(\gamma') &\rightarrow 0 \quad \text{for all } \gamma \in G. \end{aligned}$$

Lemma 3.3. *Let G be a Borel groupoid with a Borel Haar system $\{\lambda^u\}_{u \in G^{(0)}}$. Let $\{Y_i\}_{i=1}^\infty$ be a countable cover of $G^{(0)}$ by invariant Borel subsets such that each $G|_{Y_i}$ is Borel amenable. Then G is Borel amenable.*

Proof. Since each Y_i is invariant, $u \in Y_i$ implies that $(G|_{Y_i})^u = G^u$. Hence λ restricts to a Borel Haar system on $G|_{Y_i}$. Hence by [19, Proposition 2.4], the Borel amenability of $G|_{Y_i}$ implies that there is a Borel approximate invariant density, $\{g_n^i\}_{n=1}^\infty$, on $G|_{Y_i}$.

Let $B_1 = Y_1$ and if $i \geq 2$, let $B_i = Y_i \setminus \bigcup_{j=1}^{i-1} Y_j$. Then the $\{B_i\}$ are a pairwise disjoint cover of $G^{(0)}$ by invariant Borel sets (some of which might be empty). Let b^i be the characteristic function of B_i . Then each b^i is a Borel function on $G^{(0)}$

(taking the values 0 and 1) such that b_i vanishes off Y_i and for each $u \in G^{(0)}$, there is one and only one i such that $b^i(u) = 1$. In particular, we trivially have

$$\sum_{i=1}^{\infty} b^i(u) = 1 \quad \text{for all } u \in G^{(0)}.$$

For each n , define g_n on G by

$$g_n(\gamma) = \sum_i g_n^i \cdot b^i,$$

where $g_n^i \cdot b^i(\gamma) = g_n^i(\gamma) b^i(s(\gamma))$.

By invariance,

$$\int_G g_n(\gamma) d\lambda^u(\gamma) = \int_G g_n^i(\gamma) d\lambda^u(\gamma),$$

where i is such that $b^i(u) = 1$. Now it is clear that $\{g_n\}$ is a Borel approximate invariant density for G . Hence G is Borel amenable as claimed. \square

Recall that by assumption, all of our Borel spaces, and our Borel groupoids in particular, are analytic Borel spaces.

Lemma 3.4. *Let G be a Borel groupoid acting freely on (the right of) a Borel space X such that the quotient map $q : X \rightarrow X/G$ has a Borel cross section. Then X is a proper Borel G -space.*

Proof. Let $X \rtimes G$ be the transformation groupoid for the right G -action.³ The map $(x, \gamma) \mapsto (x, x \cdot \gamma)$ is a Borel as well as an algebraic isomorphism of $X \rtimes G$ onto the equivalence relation groupoid $X *_q X = \{(x, y) \in X \times X : q(x) = q(y)\}$. Since $X *_q X$ and $X \rtimes G$ are both Borel subsets of the corresponding product spaces, they are themselves analytic Borel spaces. Hence $X \rtimes G$ and $X *_q X$ are Borel isomorphic by Corollary 2 of [2, Theorem 3.3.4]. Hence it suffices to see that $X *_q X$ is proper.

We can identify X with the unit space of $X *_q X$. For each $x \in X$, let $m^x = \delta_x \times \delta_{c(q(x))}$ where $c : X/G \rightarrow X$ is our Borel cross section for q . Since $c(q(x)) = c(q(y))$ if $(x, y) \in X *_q X$, we have $(y, x) \cdot m^x = m^y$. Hence X is a proper Borel G -space. \square

4. COCYCLES

In this paper, by a *cocycle* on a groupoid G we mean a homomorphism $c : G \rightarrow Q$ into a group Q . As in [17, Definition 1.1.9],⁴ the corresponding *skew-product* groupoid is

$$G(c) = \{(a, \gamma, b) \in Q \times G \times Q : b = ac(\gamma)\}.$$

Then multiplication is given by $(a, \eta, b)(b, \gamma, d) = (a, \eta\gamma, d)$ and inversion by $(a, \gamma, b)^{-1} = (b, \gamma^{-1}, a)$. We can identify the unit space of $G(c)$ with $G^{(0)} \times Q$, and then the range and source maps are given as expected: $r(b, \gamma, a) = (r(\gamma), b)$ while $s(b, \gamma, a) = (s(\gamma), a)$. If G is a locally Hausdorff, locally compact groupoid with a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$ and if Q is a locally compact group, then provided c

³Here $X \rtimes G = \{(x, \gamma) \in X \times G : s(x) = r(\gamma)\}$ has multiplication defined by $(x, \gamma)(x \cdot \gamma, \eta) = (x, \gamma\eta)$.

⁴In [17], $G(c)$ is described as pairs $(\gamma, a) \in G \times Q$. The groupoids are isomorphic via the map $(a, \gamma, b) \mapsto (\gamma, a)$.

is continuous, $G(c)$ is a locally Hausdorff, locally compact groupoid with Haar system $\beta = \{\beta^{(u,a)}\}_{(u,a) \in G^{(0)} \times Q}$ where

$$\beta^{(u,a)}(g) = \int_G g(a, \gamma, ac(\gamma)) d\lambda^u(\gamma).$$

Later it will be useful to note that Q acts on the left of $G(c)$ by (groupoid) automorphisms:

$$q \cdot (b, \gamma, a) = (qb, \gamma, qa).$$

The action of Q on $G^{(0)} \times Q$, identified with the unit space of $G(c)$, is given by $q \cdot (u, a) = (u, qa)$.

Part of our interest in $G(c)$ is due to the following result.

Proposition 4.1 ([17, Proposition II.3.8]). *Suppose that G is a second countable locally Hausdorff, locally compact groupoid with a Haar system and that c is a continuous homomorphism from G into a locally compact group Q . If $G(c)$ and Q are both amenable, then so is G .*

Proof. By [19, Definition 2.7], we need to construct a topological approximate invariant mean $\{g_n\}$ in $\mathcal{C}(G)$. It will suffice to show that for each pair of compact sets $K \subset G$ and $L \subset G^{(0)}$, and each $\epsilon > 0$, there is a nonnegative function $f \in \mathcal{C}(G)$ such that

- (A) $\int_G f(\gamma) d\lambda^u(\gamma) \leq 1$ for all $u \in G^{(0)}$,
- (B) $\int_G f(\gamma) d\lambda^u(\gamma) \geq 1 - \epsilon$ for all $u \in L$ and
- (C) $\int_G |f(\gamma^{-1}\gamma') - f(\gamma')| d\lambda^{r(\gamma)}(\gamma') \leq \epsilon$ for all $\gamma \in K$.

Since Q is amenable and $c(K)$ is compact, there is a nonnegative function $k \in C_c(Q)$ such that with respect to a fixed *right* Haar measure α on Q we have

$$\int_Q k(a) d\alpha(a) = 1 \quad \text{and} \quad \int_Q |k(ab) - k(a)| d\alpha(a) \leq \frac{\epsilon}{2} \quad \text{for all } b \in c(K).$$

On the other hand, the (topological) amenability of $G(c)$ allows us to find a nonnegative function $g \in \mathcal{C}(G(c))$ such that

- (a) $\int_G g(a, \gamma, ac(\gamma)) d\lambda^u \leq 1$ for all $(u, a) \in G^{(0)} \times Q$,
- (b) $\int_G g(a, \gamma, ac(\gamma)) d\lambda^u \geq 1 - \epsilon$ for all $(u, a) \in L \times \text{supp } k$ and
- (c) $\int_G |g(ac(\gamma), \gamma^{-1}\gamma', ac(\gamma')) - g(a, \gamma', ac(\gamma'))| d\lambda^{r(\gamma)} \leq \epsilon$ for all $(a, \gamma) \in \text{supp } k \cdot c(K)^{-1} \times K$.

Now we define a nonnegative function on G via

$$f(\gamma) := \int_G k(a) g(a, \gamma, ac(\gamma)) d\alpha(a).$$

If g is continuous and supported in a compact Hausdorff subset, then f is too. Since in general, g a finite sum of such functions, we have $f \in \mathcal{C}(G)$. Then it is easy to check f satisfies (A) and (B) above. On the other hand,

$$\int_G |f(\gamma^{-1}\gamma') - f(\gamma')| d\lambda^{r(\gamma)}(\gamma')$$

$$\begin{aligned}
&= \int_G \int_Q k(a) (g(a, \gamma^{-1}\gamma', ac(\gamma^{-1}\gamma')) - g(a, \gamma', ac(\gamma'))) d\lambda^{r(\gamma)}(\gamma') \\
&= \int_G \int_Q k(ac(\gamma)) (g(ac(\gamma), \gamma^{-1}\gamma', a(c(\gamma'))) - g(a, \gamma', ac(\gamma'))) d\lambda^{r(\gamma)}(\gamma') d\alpha(a) \\
&\quad + \int_G \int_Q (k(ac(\gamma)) - k(a)) g(a, \gamma', ac(\gamma')) d\lambda^{r(\gamma)}(\gamma') d\alpha(a)
\end{aligned}$$

which, in view of our assumptions on k and g , is

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

Theorem 4.2. *Suppose that G is a locally Hausdorff, locally compact second countable groupoid with a Haar system and that $c : G \rightarrow Q$ is a continuous homomorphism into a locally compact group Q such that the kernel $c^{-1}(e)$ is amenable. Let $\tilde{s} : G \rightarrow G^{(0)} \times Q$ be the map given by $\tilde{s}(\gamma) = (s(\gamma), c(\gamma))$. Let Y be the image of \tilde{s} in $G^{(0)} \times Q$. Then Y is $G(c)$ -invariant. Suppose that there are countably many $q_n \in Q$ such that*

$$G^{(0)} \times Q = \bigcup_n q_n \cdot Y,$$

then $G(c)$ is amenable. In particular, if Y is open in $G^{(0)} \times Q$, then $G(c)$ is amenable. If in addition, Q is amenable, then so is G .

It is not hard to check that Y is invariant: if $s(a, \gamma, ac(\gamma)) = (s(\gamma), ac(\gamma)) \in Y$, then there is an $\eta \in G$ such that $\tilde{s}(\eta) = (s(\gamma), ac(\gamma))$. Hence $s(\eta) = s(\gamma)$ and $c(\eta) = ac(\gamma)$. But then $\tilde{s}(\eta\gamma^{-1}) = (r(\gamma), c(\eta)c(\gamma)^{-1}) = (r(\gamma), a) = r(a, \gamma, ac(\gamma))$.

Our key tool is the following observation.

Proposition 4.3. *Let G , Q , c and $Y = \tilde{s}(G)$ be as in the statement of Theorem 4.2. Then the Borel groupoids $G(c)|_Y$ and $c^{-1}(e)$ are Borel equivalent.*

Remark 4.4. The proof is complicated by the fact that we are not assuming G is Hausdorff, nor is it clear whether $c^{-1}(e)$ has a Haar system. Thus we cannot apply [1, Corollary 2.1.17] to establish that the actions are proper. Instead, we resort to the definition.

Proof. We want to show that G is a $(c^{-1}(e), G(c)|_Y)$ -Borel equivalence. We let $c^{-1}(e)$ act on the left by multiplication in G . (Hence the moment map for the left action is just r .) For the right action of $G(c)|_Y$ we use the moment map \tilde{s} . Thus $\eta \cdot (a, \gamma, ac(\gamma))$ makes sense only when $(\eta, \gamma) \in G^{(2)}$ and $c(\eta) = a$. Then we define

$$(4.1) \quad \eta \cdot (c(\eta), \gamma, c(\eta\gamma)) = \eta\gamma.$$

To see that G is a proper right $G(c)|_Y$ -space, we need to see that the transformation groupoid $G \rtimes G(c)|_Y$ for the right $G(c)|_Y$ -action is proper. But the map

$$(\eta, (c(\eta), \gamma, c(\eta\gamma))) \mapsto (\eta, \gamma)$$

is clearly a bijection of $G \rtimes G(c)|_Y$ onto $G \rtimes G$. It is easy to see that it is a groupoid homomorphism as well. Since the right action of any Borel groupoid on itself is proper (by a left to right modification of [1, Example 2.1.4(1)]), it follows that G is a proper $G(c)|_Y$ -space. The action is clearly free.

To see that the left action is proper in the Borel sense, we first note that it is certainly free and proper topologically. Hence the orbit space is itself a locally

Hausdorff, locally compact space by [13, Lemma 2.6]. Hence it is a standard Borel space with the Borel structure coming from its topology. Let $q : G \rightarrow c^{-1}(e) \backslash G$ be the quotient map. Since q is continuous and $c^{-1}(e)$ is closed, the inverse image of points in $c^{-1}(e) \backslash G$ under q are closed. Since open sets in G are σ -compact, the forward image of open sets are σ -compact. Since a compact subset of a locally Hausdorff, locally compact space is Borel, so is the forward image of any open set. Hence q has a Borel cross section w by [2, Theorem 3.4.1]. *A priori* $c^{-1}(e) \backslash G$ has two Borel structures: the one we've been using coming from the quotient topology, and also the quotient Borel structure. The quotient map q is Borel in both cases and w must also be a cross section for the potentially finer quotient Borel structure. Hence both spaces are Borel isomorphic to the image of w by [2, Proposition 3.4.2]. Hence the two Borel structures coincide on the orbit space. Therefore Lemma 3.4 applies and G is a free and proper Borel $c^{-1}(e)$ -space as required.

If $\eta \in c^{-1}(e)$, then $\tilde{s}(\eta\gamma) = (s(\gamma), c(\eta\gamma)) = (s(\gamma), c(\gamma))$. It follows that \tilde{s} factors through $G/c^{-1}(e)$. On the other hand, suppose that

$$\tilde{s}(\gamma) = (s(\gamma), c(\gamma)) = \tilde{s}(\eta) = (s(\eta), c(\eta)).$$

Then $c(\gamma) = c(\eta)$ and $\gamma^{-1}\eta \in c^{-1}(e)$. Of course $\gamma \cdot \gamma^{-1}\eta = \eta$. Thus \tilde{s} induces a bijection of $G/c^{-1}(e)$ with Y . Since $c^{-1}(e)$ is closed, it acts freely and properly on G . Thus the quotient $G/c^{-1}(e)$ is a standard Borel space (with respect to the quotient Borel structure) and \tilde{s} induces a Borel bijection of $G/c^{-1}(e)$ with the analytic Borel space Y . Hence \tilde{s} is a Borel isomorphism by [2, Corollary 2 of Theorem 3.3.4].

It is clear from (4.1) that $\tilde{r} = r$ factors through $G/G(c)|_Y$. In fact, the homeomorphism of $G \rtimes G(c)|_Y$ with $G \rtimes G$ induces a homeomorphism of $G/G(c)|_Y$ onto $G/G \cong G^{(0)}$. Hence \tilde{r} certainly induces a Borel isomorphism as required. \square

Proof of Theorem 4.2. If Y is open, then $\{q \cdot Y\}_{q \in Q}$ is an open cover of the second countable set $G^{(0)} \times Q$. By a theorem of Lindelöf, it has a countable subcover; so it suffices to prove the first statement.

Since G is σ -compact and \tilde{s} is continuous, Y is also σ -compact and hence Borel. This ensures that $G(c)|_Y$ is standard as a Borel space. Since $c^{-1}(e)$ is assumed to be amenable, it is Borel amenable. Hence $G(c)|_Y$ is Borel amenable by Proposition 4.3. On the other hand $G(c)|_{q_n \cdot Y} = q_n \cdot (G|_Y)$. Hence each $G(c)|_{q_n \cdot Y}$ is Borel amenable. Since Q acts by automorphisms, each $q_n \cdot Y$ is invariant. Now $G(c)$ is Borel amenable from Lemma 3.3. Since $G(c)$ has a Haar system if G does, it follows from [19, Corollary 2.15] that G is amenable.

The last assertion follows from Proposition 4.1. \square

As special cases of Theorem 4.2, we obtain the cocycle results from [1] and [22] mentioned in the introduction. The following result strengthens [22, Theorem 9.3] by removing the hypotheses that G be étale.

Corollary 4.5. *Suppose that G is a locally Hausdorff, locally compact second countable groupoid with a Haar system and that $c : G \rightarrow Q$ is a continuous homomorphism into a discrete group Q such that $c^{-1}(e)$ is amenable. Then $G(c)$ is amenable. If Q is amenable, then so is G .*

Proof. It suffices to see that $Y = \tilde{s}(G)$ is open in $G^{(0)} \times Q$. But s is open and

$$\tilde{s}(G) = \bigcup_{a \in Q} s(c^{-1}(a)) \times \{a\},$$

which is open since each $c^{-1}(a)$ is open. \square

The result [1, Theorem 5.3.14] mentioned in the introduction establishes the amenability of a measured groupoid G which admits a strongly surjective Borel homomorphism onto a Borel groupoid with amenable kernel and range. The following result is similar, but concerns topological instead of measure amenability. Moreover, it is more restrictive since it assumes that the range is a group rather than a groupoid.

Corollary 4.6. *Suppose that G is a locally Hausdorff, locally compact second countable groupoid with a Haar system and that $c : G \rightarrow Q$ is a strongly surjective continuous homomorphism into a group Q such that $c^{-1}(e)$ is amenable. Then $G(c)$ is amenable. If Q is amenable, then so is G .*

Proof. Since c is strongly surjective, we must have $c(G_u) = Q$ for all u . It follows that \tilde{s} is surjective, and the result follows from Theorem 4.2. \square

5. APPLICATION TO SEMIGROUP ACTIONS

Our results apply nicely to semigroup action groupoids.

Definition 5.1. Let X be a set and P be a semigroup with identity e . A *right (partial) action of P on X* consists of a subset $X * P$ of $X \times P$ and a map $T : X * P \rightarrow X$ sending (x, m) to $x \cdot m$, such that

- (a) for all $x \in X$, $(x, e) \in X * P$ and $x \cdot e = x$;
- (b) for all $(x, m, n) \in X \times P \times P$, $(x, mn) \in X * P$ if and only if $(x, m) \in X * P$ and $(x \cdot m, n) \in X * P$; if this holds, $(x \cdot m) \cdot n = x \cdot (mn)$.

For all $m \in P$, we define $U(m) = \{x : (x, m) \in X * P\}$ and $V(m) = \{x \cdot m : (x, m) \in X * P\}$ and $T_m : U(m) \rightarrow V(m)$ such that $T_m x = x \cdot m$. The triple (X, P, T) will be called a *semigroup action*.

This notion generalizes that of singly generated dynamical system (SGDS) in the sense of [18], which is the case $P = \mathbf{N}$. A SGDS is given by a single map T from a subset $\text{dom}(T)$ of X to another subset $\text{ran}(T)$ of X . For $n \in \mathbf{N}$, we let $U(n) = \text{dom}(T^n)$ and define $x \cdot n = T^n(x)$ for $x \in U(n)$.

Let us define the binary relation on P : $m \leq m'$ if and only if there exists $n \in P$ such that $m' = mn$. Our axioms imply that $m \leq m' \Rightarrow U(m') \subset U(m)$.

The next notion is important because it will allow a simple construction of a groupoid from a semigroup action.

Definition 5.2. Let us say that a semigroup action (X, P, T) is *directed* if for all pairs $(m, n) \in P \times P$ such that $U(m) \cap U(n)$ is non-empty, there exists an upper bound $r \in P$ of m and n such that $U(m) \cap U(n) = U(r)$.

Note that the equality can be replaced by the inclusion $U(m) \cap U(n) \subset U(r)$. Here are two rather different situations where the action is directed.

Example 5.3. Assume that the action is everywhere defined, i.e. $X * P = X \times P$. Then the action is directed if and only if the semigroup P is directed, in the sense that for all $m, n \in P$, there exists $r \in P$ such that $m, n \leq r$; we then say that r is

a common upper bound, or c.u.b. for short, of m and n . If P is a subsemigroup of a group Q , meaning that

$$(5.1) \quad P \subset Q, \quad PP \subset P \quad \text{and} \quad e \in P.$$

then, P is directed if and only if it satisfies the *Ore condition* $P^{-1}P \subset PP^{-1}$. In such a case, P is called an *Ore semigroup*.

Example 5.4. Assume that P is *quasi-lattice ordered*. This means that P is a subsemigroup of a group Q such that $P \cap P^{-1} = \{e\}$ and whenever two elements $m, n \in P$ have a c.u.b. in P , they have a least upper bound, or l.u.b. for short, $m \vee n$ (the original definition of A. Nica in [14] contains the additional assumption that every element of Q which has an upper bound in P has a least upper bound in P). Then, the action of P on $X = P$ given by

$$U(n) = \{x \in P : n \leq x\} \quad \text{and} \quad x \cdot n = n^{-1}x$$

is directed.

If (X, P, T) is directed, then the relation $x \sim y$ if and only if there exist $(m, n) \in P \times P$ such that $x \cdot m = y \cdot n$ is transitive and hence an equivalence relation. We denote by $[x]$ the equivalence class of x .

Given P a subsemigroup of a group Q and a directed semigroup action (X, P, T) , we define $G(X, P, T)$ as the set of triples (x, q, y) in $X \times Q \times X$ such that there exist $m, n \in P$ with $q = mn^{-1}$, $x \in U(m)$, $y \in U(n)$ and $x \cdot m = y \cdot n$.

Lemma 5.5. *Assume that P is a subsemigroup of a group Q and that (X, P, T) is a directed action. Then $G(X, P, T)$ is a subgroupoid of $X \times Q \times X$, equipped with its natural structure of a groupoid over X .*

Proof. Suppose that (x, q, y) belongs to G : there exist $m, n \in P$ such that $q = mn^{-1}$ and $x \cdot m = y \cdot n$. Then (y, q^{-1}, x) also belongs to G because $q^{-1} = nm^{-1}$ and $y \cdot n = x \cdot m$. Suppose that (x, s, y) and (y, t, z) belong to G : there exist $m, n, p, q \in P$ such that $s = mn^{-1}$, $t = pq^{-1}$, $x \cdot m = y \cdot n$ and $y \cdot p = z \cdot q$. Since the action is directed, there exist $(a, b) \in P \times P$ such that $na = pb$. Then $x \cdot ma = y \cdot na = y \cdot pb = z \cdot qb$. Since $(ma)(qb)^{-1} = (mn^{-1})(pq^{-1})$, (x, st, y) belongs to G . \square

Definition 5.6. Let (X, P, T) be a directed semigroup action, where P is a subsemigroup of a group Q . The groupoid $G(X, P, T)$ associated with (X, P, T) is called the *semidirect product groupoid* (or *semidirect product* for short) of the action. It carries the canonical cocycle $c : G(X, P, T) \rightarrow Q$ given by $c(x, q, y) = q$.

Remark 5.7. When X is reduced to a point, the lemma says that PP^{-1} is a group if P satisfies the Ore condition $P^{-1}P \subset PP^{-1}$; clearly, this condition is also necessary.

We next make the following topological assumptions:

Definition 5.8. We shall say that a semigroup action (X, P, T) , where P is a subsemigroup of a group Q , is *locally compact* if

- (a) X is a locally compact Hausdorff space;
- (b) Q is a discrete group;
- (c) for all $m \in P$, $U(m)$ and $V(m)$ are open subsets of X and $T_m : U(m) \rightarrow V(m)$ is a local homeomorphism.

In the following, we always assume that the semigroup action (X, P, T) is locally compact.

Given $m, n \in P$ and A, B subsets of X , we define

$$Z(A, m, n, B) := \{ (x, mn^{-1}, y) \in G : x \in A, y \in B \text{ and } x \cdot m = y \cdot n \}$$

Let \mathcal{B} be the family of subsets $Z(U, m, n, V)$, where U and V are open subsets of X .

Lemma 5.9. *The family \mathcal{B} is a base for a topology \mathcal{T} on G .*

Proof. First, it is clear that this family covers G . Second, let (x, q, y) be a point in the intersection of $Z(U, m, n, V)$ and $Z(U', m', n', V')$. We are going to find $Z(U'', m'', n'', V'')$ containing the point (x, q, y) and contained in the intersection. There exists $(a, a') \in P \times P$ such that $ma = m'a'$ is an upper bound r of m and m' and $U(r) = U(m) \cap U(m')$. Then $na = n'a'$ is an upper bound s of n and n' and $U(s) = U(n) \cap U(n')$. Let W be an open neighborhood of $x \cdot m = y \cdot n$ on which T_a is injective. Similarly, let W' be an open neighborhood of $x \cdot m' = y \cdot n'$ on which $T_{a'}$ is injective. We let $U'' = U \cap U' \cap T_m^{-1}(W) \cap T_{m'}^{-1}(W')$, $V'' = V \cap V' \cap T_n^{-1}(W) \cap T_{n'}^{-1}(W')$, $m'' = r$ and $n'' = s$. Since $x \in U''$, $y \in V''$ and $x \cdot r = y \cdot s$, (x, q, y) belongs to $Z(U'', m'', n'', V'')$. If (x', q, y') belongs to $Z(U'', m'', n'', V'')$, $x' \in U \cap U'$, $y' \in V \cap V'$ and $x' \cdot ma = y' \cdot na$. Since $x' \cdot m$ and $y' \cdot n$ belong to W on which T_a is injective, $x' \cdot m = y' \cdot n$. Therefore (x', q, y') belongs to $Z(U, m, n, V)$. Similarly, the equality $x' \cdot m'a' = y' \cdot n'a'$ implies the equality $x' \cdot m' = y' \cdot n'$ because $x' \cdot m'$ and $y' \cdot n'$ belong to W' on which $T_{a'}$ is injective. \square

Remark 5.10. The sets $Z(U, m, n, V)$, where U and V are open subsets of X such that $U \subset U(m)$, $V \subset U(n)$ and $T_{m|U}$ and $T_{n|V}$ injective form a subbase of \mathcal{B} .

Lemma 5.11. *The topology \mathcal{T} of G is finer than the product topology of $X \times Q \times X$ but it agrees with it on the sets $Z(A, m, n, B)$, where A, B are subsets of X and $m, n \in P$.*

Proof. The intersection of a rectangle $U \times \{q\} \times V$ with G , where U, V are open subsets of X is a union of elements of \mathcal{B} . Therefore, the topology \mathcal{T} is finer than the product topology. Let A, B be subsets of X and $m, n \in P$. Let (x, q, y) be a point of $Z(A, m, n, B)$. Let $Z(U, m', n', V) \cap Z(A, m, n, B)$ be a basic neighborhood of (x, q, y) in $Z(A, m, n, B)$. Just as in the proof of the previous lemma, we introduce $(a, a') \in P \times P$ such that $ma = m'a'$ (denoted by r) and $U(r) = U(m) \cap U(m')$, $s = na = n'a'$, and we choose an open neighborhood W of $x \cdot m = y \cdot n$ on which T_a is injective, and an open neighborhood W' of $x \cdot m' = y \cdot n'$ on which $T_{a'}$ is injective. We define $U' = U \cap T_m^{-1}(W) \cap T_{m'}^{-1}(W')$ and $V' = V \cap T_n^{-1}(W) \cap T_{n'}^{-1}(W')$. Suppose that (x', q, y') belongs to $(U' \times \{q\} \times V') \cap Z(A, m, n, B)$. Then $x' \cdot m'a' = x' \cdot ma = y' \cdot na = y' \cdot n'a'$. We deduce as before that $x' \cdot m' = y' \cdot n'$. Therefore (x', q, y') belongs to $Z(U, m', n', V)$. \square

Proposition 5.12. *Let (X, P, T) be a semigroup action as above. We endow $G(X, P, T)$ with the topology \mathcal{T} . Then,*

- (a) $G(X, P, T)$ is an étale locally compact Hausdorff groupoid;
- (b) the canonical cocycle $c : G(X, P, T) \rightarrow Q$ is continuous.

Proof. The topology \mathcal{T} is Hausdorff because it is finer than the product topology. Let $(x, q, y) \in G$. Pick $(m, n) \in P \times P$ such that $q = mn^{-1}$ and $x \cdot m = y \cdot n$. Let A, B be compact neighborhoods of x, y contained respectively in $U(m)$ and in $U(n)$. Then $Z(A, m, n, B)$ is a compact neighborhood of (x, q, y) , and \mathcal{T} is a locally compact topology. The injection map $i(x) = (x, e, x)$ is a homeomorphism from X onto $G^{(0)}$.

The inverse map $(x, q, y) \mapsto (y, q^{-1}, x)$ transforms $Z(U, m, n, V)$ into $Z(V, n, m, U)$. Therefore, it is a homeomorphism. Suppose that $(x_\alpha, s_\alpha, y_\alpha)$ converges to (x, s, y) and $(y_\alpha, t_\alpha, z_\alpha)$ converges to (y, t, z) . Pick basic open sets $Z(U, m, n, V)$ and $Z(V', p, q, W)$ containing respectively (x, s, y) and (y, t, z) . By definition, for α large enough $(x_\alpha, s_\alpha, y_\alpha)$ belongs to $Z(U, m, n, V)$ and $(y_\alpha, t_\alpha, z_\alpha)$ belongs to $Z(V', p, q, W)$. As in Lemma 5.11, pick $(a, b) \in P \times P$ such that $na = pb$. Then (x, st, z) belongs to $Z(U, ma, qb, W)$ and so does $(x_\alpha, s_\alpha t_\alpha, z_\alpha)$ for α large enough. We apply Lemma 5.11 to conclude that $(x_\alpha, s_\alpha t_\alpha, z_\alpha)$ converges to (x, st, z) . Thus G is a topological locally compact Hausdorff groupoid.

A basis element $S = Z(U, m, n, V)$ such that T_m is injective on U and T_n is injective on V is a bisection, in the sense that the restrictions of the range and source maps $r|_S$ and $s|_S$ are homeomorphisms onto open subsets of X . Since G admits a cover of open bisections, it is an étale groupoid.

Since the canonical cocycle is continuous with respect to the product topology, it is continuous with respect to \mathcal{T} . \square

Here is our main application of our Theorem 4.2 (in the form of Corollary 4.5); it gives the amenability of the semidirect product by a subsemigroup of an amenable group.

Theorem 5.13. *Let (X, P, T) be a directed locally compact semigroup action. Assume that P is a subsemigroup of a countable amenable group Q . Then the semidirect product groupoid $G(X, P, T)$ is topologically amenable.*

Proof. To prove this result, we apply Corollary 4.5 to the continuous cocycle $c : G(X, P, T) \rightarrow Q$. The only missing point is the amenability of the equivalence relation $R = c^{-1}(e)$. As is standard, we try to write R as the increasing union of well-behaved equivalence relations (for example, see [21, Lemma 3.5]).

To see how to do this, we call a subset $F \subset P$ *action-directed* if $e \in F$ and given $n, m \in F$ with $U(n) \cap U(m) \neq \emptyset$, then there is an $r \in F$ dominating n and m such that $U(r) = U(n) \cap U(m)$. For example, by hypotheses, P itself is action-directed. More to the point, if F is action-directed, then

$$R_F = \{ (x, y) : \text{there is a } m \in F \text{ such that } x \cdot m = y \cdot m \}$$

is a Borel equivalence relation: it is an F_σ subset of $X \times X$ and an equivalence relation since F is action-directed. We just need to specify suitable sets F .

Let \mathcal{F} be the collection of finite subsets F of P such that $\bigcap_{m \in F} U(m) \neq \emptyset$. A simple induction argument implies the following.

Claim 1. If $F \in \mathcal{F}$, then there is an $r \in P$ such that $n \leq r$ for all $n \in F$ and $U(r) = \bigcap_{n \in F} U(n)$.

Claim 2. There is a map $F \mapsto r_F$ from \mathcal{F} to P such that $r_\emptyset = e$, $r_{\{n\}} = n$, and such that

- (a) $n \leq r_F$ for all $n \in F$,

- (b) $U(r_F) = \bigcap_{n \in F} U(n)$, and
- (c) $F' \subset F$ implies $r_{F'} \leq r_F$.

Proof of Claim 2. We start by defining r_\emptyset and $r_{\{n\}}$ as above. Suppose that we have defined r_F for all $F \in \mathcal{F}$ with $|F| \leq k$ for some $k \geq 1$ such that (a), (b) and (c) hold.

Let us define r_F for $F \in \mathcal{F}$ with $k+1$ elements. Note that

$$\mathcal{F}' = \{ S \subset F : |S| = k \}$$

is a subset of \mathcal{F} . By Claim 1, we can define $r_S \in P$ so that $r_S \leq r_F$ for all $S \in \mathcal{F}'$ and such that $U(r_S) = \bigcap_{n \in S} U(n)$.

Now consider the set $\{ r_F : F \in \mathcal{F} \text{ and } |F| \leq k+1 \}$. Then (a) and (b) hold by assumption if $|F| \leq k$. But if $|F| = k+1$ and $n \in F$, then there is an $S \in \mathcal{F}'$ such that $n \in S$. Hence $n \leq r_S \leq r_F$. Similarly,

$$U(r_F) = \bigcap_{S \in \mathcal{F}'} U(r_S) = \bigcap_{S \in \mathcal{F}'} \bigcap_{n \in S} U(n) = \bigcap_{n \in F} U(n).$$

Hence (a) and (b) hold for sets of $k+1$ or fewer elements.

Now suppose F' is a proper subset of F . We have $r_{F'} \leq r_F$ by assumption if $|F'| \leq k$. If $|F'| = k+1$, then there exists $S \subset F$ with $|S| = k$. Then $r_{F'} \leq r_S \leq r_F$. \square

As pointed out by the referee, the possibility that \mathcal{F} might be directed is suggested by [10] where the authors work with quasi-lattice ordered subgroups. That the same is true for directed actions is implied by the following claim.

Claim 3. Every finite subset $S \subset P$ is contained in a *finite* action-directed subset F .

Proof of Claim 3. Let $F \mapsto r_F$ be the map defined in Claim 2. Define

$$F = \{ r_{S'} : S' \subset S \text{ and } S' \in \mathcal{F} \}.$$

Clearly $e \in F$ and $S \subset F$. We claim that F is action-directed. Let $k, l \in F$ be such that $U(k) \cap U(l) \neq \emptyset$. We can assume that $k = r_{S'}$ and $l = r_{S''}$ for appropriate subsets of S . Then $F' = S' \cup S''$ satisfies $\bigcap_{n \in F'} U(n) = U(k) \cap U(l) \neq \emptyset$. But then $r_{F'} \in F$. Since $F \mapsto r_F$ is monotonic, we have $r_{S'} \leq r_{F'}$ and $r_{S''} \leq r_{F'}$. This completes the proof of the claim. \square

Claim 4. There is a sequence (F_i) of finite action-directed sets such that $F_i \subset F_{i+1}$ and such that

$$c^{-1}(e) = \bigcup_i R_{F_i}.$$

Proof of Claim 4. Let $P = \{p_1, p_2, \dots\}$ with $p_1 = e$. Now we can employ Claim 3 to inductively construct the F_i where F_{i+1} is an action-directed set containing F_i and p_{i+1} . \square

The key observation is that if F is finite and action-directed, then R_F is proper. To see this, note that the equivalence class $[x]_F$ is the finite union over $m \in F$ of the fibres $T_m^{-1}(x \cdot m)$. Since T_m is a local homeomorphism, the fibres are discrete. Hence each orbit is discrete and therefore locally closed. The Mackey-Glimm-Ramsay dichotomy [16] then implies that the orbit space X/R_F is a standard Borel space. Then [1, Example 2.1.4(2)] implies that R_F is a proper Borel groupoid.

It follows from Claim 4 and Lemma 2.8 that $c^{-1}(e)$ is Borel amenable. Since it is open in $G(X, P, T)$, it is étale. Hence it is amenable by [19, Corollary 2.15]. \square

Remark 5.14. It is well known that there are interesting amenable actions (in the sense that the semi-direct product groupoid of the action is amenable) of non-amenable groups. We will encounter in the next section an amenable action of a free semigroup (appearing in the work of Nica [14] on the Wiener-Hopf algebra of a semigroup). In this case, the amenability of the action cannot be deduced from the above theorem as the free group is not amenable.

6. APPLICATION TO TOPOLOGICAL HIGHER RANK GRAPHS

Higher-rank graphs provide interesting semigroup actions which generalize one-sided subshifts of finite type. We recall some definitions but refer to [26] for a complete exposition.

We introduce two changes with respect to [26]. First, we define P -graphs for an arbitrary subsemigroup P of a group Q while Yeend considers the case $P = \mathbb{N}^d \subset Q = \mathbb{Z}^d$. In order to develop the theory smoothly, we shall often need the assumption that P is quasi-lattice ordered, as defined in Example 5.4. Such P -graphs have already been introduced in [4]. Second, we construct the path space Ω as a closure in the space of closed subsets of the P -graph with respect to the Fell topology. The use of directed hereditary subsets to construct the path space goes back to [14] and is present in [4, 8, 22].

Here are our definitions. A *small category* Λ is given by its set of arrows $\Lambda^{(1)}$ (usually denoted by Λ), its set of vertices $\Lambda^{(0)}$ (viewed as a subset of Λ through the identity map $i : \Lambda^{(0)} \rightarrow \Lambda$), range and source maps $r, s : \Lambda \rightarrow \Lambda^{(0)}$ and composition map $\circ : \Lambda^{(2)} \rightarrow \Lambda$ where $\Lambda^{(2)}$ is the set of composable pairs of arrows, i.e., $(\lambda, \mu) \in \Lambda \times \Lambda$ such that $s(\lambda) = r(\mu)$. Given $A, B \subset \Lambda$, we write $A * B = (A \times B) \cap \Lambda^{(2)}$. We make the following topological assumptions.

- (a) Λ and $\Lambda^{(0)}$ are locally compact Hausdorff spaces;
- (b) $r, s : \Lambda \rightarrow \Lambda^{(0)}$ are continuous and s is a local homeomorphism;
- (c) $i : \Lambda^{(0)} \rightarrow \Lambda$ is continuous;
- (d) composition $\circ : \Lambda^{(2)} \rightarrow \Lambda$ is continuous and open.

The following definition is a topological version of [4, Definition 2.1].

Definition 6.1. Let P be a semigroup with unit element e . A *higher-rank topological graph graded by P* , or *P -graph* for short, is a topological small category Λ as above endowed with a map, called the degree map, $d : \Lambda \rightarrow P$ which satisfies the following properties

- (a) the degree map $d : \Lambda \rightarrow P$ is continuous (where P has the discrete topology);
- (b) for all $(\mu, \nu) \in \Lambda^{(2)}$, $d(\mu\nu) = d(\mu)d(\nu)$ and for all $v \in \Lambda^{(0)}$, $d(v) = e$;
- (c) it has the unique factorization property: for all $m, n \in P$, the composition map $\Lambda^m * \Lambda^n \rightarrow \Lambda^{mn}$ is a homeomorphism.

As a basic example of P -graph, we consider the graph of a semigroup action. If (X, P, T) is a locally compact semigroup action as in the previous section, set

$$\Lambda^{(0)} = X, \quad \Lambda = X * P, \quad r(x, n) = x, \quad s(x, n) = x \cdot n \quad \text{and} \quad d(x, n) = n.$$

Composition is necessarily given by $(x, m)(x \cdot m, n) = (x, mn)$. It results from our axioms of a semigroup action that Λ is a P -graph which we call the *graph of the*

action. For example, if (X, T) is a singly generated dynamical system as defined in the previous section, it is useful to think of $(x, n) \in \Lambda = X * \mathbf{N}$ as a finite path $(x, Tx, \dots, T^{n-1}x)$ when $n \geq 1$. More generally, we think of $(x, n) \in X * P$ as a finite path $\{(x \cdot m) : m \leq n\}$, although this latter set need not be finite.

Of course, not all P -graphs arise as graphs of a semigroup action. The topological \mathbf{N} -graphs are exactly the usual topological graphs. A topological graph is given by a pair of locally compact Hausdorff spaces (E, V) with two maps $r, s : E \rightarrow V$, r continuous and s local homeomorphism. The space Λ of finite paths is the disjoint union over \mathbf{N} of the spaces $E^{(n)}$ of paths of length n , where $E^{(0)} = V$, $E^{(1)} = E$ and

$$E^{(n)} = \{e_1 e_2 \dots e_n : e_i \in E, s(e_i) = r(e_{i+1}) \text{ for } i = 1, \dots, n-1\}.$$

endowed with the product topology, for $n \geq 2$. It is a topological category with $\Lambda^{(0)} = V$, the obvious range and source maps and composition given by concatenation. It has an obvious degree map $d : \Lambda \rightarrow \mathbf{N}$ where $d(\lambda) = n$ if and only if $\lambda \in E^{(n)}$. This definition includes the graphs which appear in the theory of graph C^* -algebras, which is the case when E and V are discrete spaces. A singly generated dynamical system (X, T) can be viewed as a topological graph with $V = X$, $E = \text{dom}(T)$, $r(x) = x$ and $s(x) = Tx$. Its space of finite paths Λ agrees with $X * \mathbf{N}$. As another example of topological graph, consider $(E = \mathbf{T}, V = \mathbf{T})$ where \mathbf{T} is the circle $|z| = 1$ and the range and source maps are respectively $z \mapsto z^2$ and $z \mapsto z^3$. Topological graphs where the range and source maps are both local homeomorphisms are called polymorphisms in [3].

By analogy with the above examples, the elements of a higher-rank graph Λ are called finite paths. We define $\mu \leq \lambda$ if there exists ν such that $\lambda = \mu\nu$. This is a pre-order relation which shares some of the properties of the pre-order relation we have defined on the semigroup P in the previous section. In particular, suppose that λ and μ have a common upper bound ν . Then $d(\lambda)$ and $d(\mu)$ have $d(\nu)$ as a common upper bound. If P is quasi-lattice ordered, $d(\lambda)$ and $d(\mu)$ have a least common upper bound p . Therefore, there exists $\nu' \leq \nu$ c.u.b. of λ and μ with $d(\nu') = p$. We say that ν' is a l.u.b. of λ and μ . Such a l.u.b. need not be unique. Given A, B subsets of Λ , we denote by $A \vee B$ the set of elements which are l.u.b. of some pair $(\lambda, \mu) \in A \times B$. Given $\mu \leq \lambda$, we define the segment $[\mu, \lambda] := \{\nu : \mu \leq \nu \leq \lambda\}$. We shall use the following notation. If $\lambda \in \Lambda$ and $n \in P$ are such that $n \leq d(\lambda)$, then λ can be written uniquely $\lambda = \mu\nu$ where $d(\mu) = n$. Then we define $\lambda \cdot n := \nu$. Conversely, given $\mu \in \Lambda$, we can define $\mu\nu$ for all $\nu \in r^{-1}(s(\mu))$.

We shall need some further assumptions on our higher-rank graphs.

Definition 6.2. One says that the P -graph Λ is

- (a) *(r, d)-proper* if the map $(r, d) : \Lambda \rightarrow \Lambda^{(0)} \times P$ is proper;
- (b) *compactly aligned* if P is quasi-lattice ordered and for all compact subsets $A, B \subset \Lambda$, the subset $A \vee B$ is compact.

In the setting of a discrete \mathbf{N} -graph, condition (a) means that a vertex emits finitely many edges while condition (b) is always satisfied. The graph of the action of a semigroup always satisfies (a) since (r, d) is the injection of $\Lambda = X * P$ into $X \times P$. When P is quasi-lattice ordered, condition (a) implies condition (b). Indeed, if ν belongs to $A \vee B$ where A, B are subsets of Λ , then $r(\nu)$ belongs to $r(A) \cap r(B)$ and $d(\nu)$ belongs to $d(A) \vee d(B)$. If A and B are compact, $r(A) \cap r(B)$ is compact and $d(A) \vee d(B)$ is finite.

Let Λ be a P -graph. Set

$$\Lambda * P = \{ (\lambda, m) \in \Lambda \times P : m \leq d(\lambda) \}$$

and define

$$T : \Lambda * P \rightarrow \Lambda$$

by $T(\lambda, m) := \lambda \cdot m = \nu$ if $d(\lambda) = mn$ and $\lambda = \mu\nu$ with $d(\mu) = m$ and $d(\nu) = n$.

Proposition 6.3. *Let Λ be a P -graph. Define T as above. Then T is a directed action of P on Λ by partial local homeomorphisms.*

Proof. The domain of T_m is the open set $U(m) = \{ \lambda \in \Lambda : m \leq d(\lambda) \}$. Its range

$$V(m) = \{ \nu \in \Lambda : \text{there exists } \mu \in \Lambda^m \text{ such that } (\mu, \nu) \in \Lambda^{(2)} \} = r^{-1}(s(\Lambda^m))$$

is also open because s is open. Let us show that T_m is continuous and open. Since $U(m)$ is the union of the open subsets Λ^{ma} when a runs over P , it suffices to study its restriction to Λ^{ma} . This restriction factors as

$$\Lambda^{ma} \rightarrow \Lambda^m * \Lambda^a \rightarrow \Lambda$$

The first map is a homeomorphism by assumption. The second map is the restriction of the projection onto the second factor, and is open because s is open. Therefore the composition is continuous and open. Moreover $s : \Lambda \rightarrow \Lambda^{(0)}$ is a local homeomorphism. On an open subset U of Λ on which s is injective,

$$T_{m|U} : U(m) \cap U \rightarrow \Lambda$$

is a homeomorphism onto an open subset.

To see that the action is directed, suppose that $U(m) \cap U(n) \neq \emptyset$. Then $m, n \leq d(\lambda)$ for some $\lambda \in U(m) \cap U(n)$. Then $U(m) \cap U(n) \subset U(d(\lambda))$. \square

If P is a subsemigroup of a discrete group Q and Λ is a P -graph, then we can form the semi-direct product groupoid $G(\Lambda, P, T)$. This groupoid is proper, which means that the map $(r, s) : G(\Lambda, P, T) \rightarrow \Lambda \times \Lambda$ is proper. The fine structure of such groupoids can be interesting (in the group case, see for example [6, 7]). In the theory of graph algebras, Λ is the space of finite paths. It is fruitful (and necessary) to construct a larger space, which is called the path space and which includes infinite paths. This is what we do in the next subsection.

6.1. The path space Ω . Just as in the case of a graph, we want to define a space of paths, both finite and infinite. Our construction is directly inspired by a construction of A. Nica in [14] which we recall below. In fact, our construction agrees with Nica's when $\Lambda = X * P$ and when X is reduced to one point. The same idea of defining paths as hereditary directed subsets of the graph appears also in [4, Section 3].

Let us first summarize the exposition given in [14]. There P is a subsemigroup of a discrete group Q and it is assumed to be quasi-lattice ordered. One embeds P into the space $\{0, 1\}^P$ of all subsets of P endowed with the product topology by sending $m \in P$ to the segment $j(m) = [e, m]$. The Wiener-Hopf closure of P is the closure of $j(P)$ in $\{0, 1\}^P$. It is denoted by $\Omega(P)$. Nica remarks that the elements of $\Omega(P)$ are exactly the non-empty hereditary and directed subsets of P . As we will see below, a similar construction can be employed to define a closure Ω of a topological P -graph Λ .

Recall that we have defined on Λ the pre-order $\mu \leq \lambda$ if there exists ν such that $\lambda = \mu\nu$. Note that this implies that $r(\mu) = r(\lambda)$ and $d(\mu) \leq d(\lambda)$.

Following [14], we call a subset A of Λ *hereditary* if $\mu \leq \lambda \in A$ implies $\mu \in A$, and *directed* if any two elements of A have a c.u.b. in A . Hereditary and directed subsets are called filters in [4, 8]. Our terminology is the same as in [22]. We denote by $\tilde{\Omega}$ the set of all hereditary and directed closed subsets of Λ and by Ω the set of all non-empty hereditary and directed closed subsets of Λ . We view $\tilde{\Omega}$ and Ω as subsets of the space $\mathcal{C}(\Lambda)$ of all closed subsets of Λ endowed with the Fell topology [25, §H.1]. Recall that a basis for the Fell topology on $\mathcal{C}(\Lambda)$ is given by sets of the form

$$\mathcal{U}(K; U_1, \dots, U_m) = \{ F \in \mathcal{C}(\Lambda) : F \cap K = \emptyset \text{ and } F \cap U_i \neq \emptyset \}$$

where $K \subset \Lambda$ is compact and each $U_i \subset \Lambda$ is open. Then as in [25, Lemma H.2]⁵ a net (F_β) converges to F in $\mathcal{C}(\Lambda)$ if and only if every subnet (F_i) of (F_β) is such that

- (F1) given $\lambda_i \in F_i$ such that $\lambda_i \rightarrow \lambda$, then $\lambda \in F$, and
- (F2) if $\lambda \in F$, then there is a subnet (F_{i_j}) and $\lambda_j \in F_{i_j}$ such that $\lambda_j \rightarrow \lambda$.

Lemma 6.4. (a) *Let A be a non-empty directed subset of Λ . Then A is contained in $x\Lambda := r^{-1}(x)$ for some (necessarily unique) $x \in \Lambda^{(0)}$, which will be written $r(A)$;*

(b) *the map $r : \Omega \rightarrow \Lambda^{(0)}$ is continuous.*

Proof. (a) By definition, the relation $\mu \leq \lambda$ implies that $r(\mu) = r(\lambda)$. Let $\mu, \nu \in A$. Since there exists λ such that $\mu, \nu \leq \lambda$, we must have $r(\mu) = r(\nu)$.

(b) Suppose that $A_\alpha, A \in \Omega$ and A_α tends to A . Let $x_\alpha = r(A_\alpha)$ and $x = r(A)$. Let U be an open neighborhood of x . Since $A \cap r^{-1}(U) \neq \emptyset$, there exists α_0 such that $A_\alpha \cap r^{-1}(U) \neq \emptyset$ for all $\alpha \geq \alpha_0$. Then, $x_\alpha \in U$. \square

Lemma 6.5. *Assume (r, d) is proper and that P is contained in a group Q . If λ_i converges to λ , μ_i converges to μ and for all i , $\mu_i \leq \lambda_i$, then $\mu \leq \lambda$.*

Proof. There exists a net ν_i such that $\lambda_i = \mu_i \nu_i$. Since $d(\lambda_i) = d(\mu_i)d(\nu_i)$, $d(\nu_i)$ is eventually constant. Since $r(\nu_i) = s(\mu_i)$, it is contained in some compact subset of $\Lambda^{(0)}$. Because of (r, d) -properness, there is a subnet (ν_i) converging to some ν . Then $\lambda = \mu\nu$. \square

Lemma 6.6. *Let A be a subset of Λ .*

- (a) *If A is directed (resp., hereditary), then $d(A)$ is directed (resp., hereditary).*
- (b) *If A is directed, the restriction to A of the degree map $d|_A : A \rightarrow P$ is a bijection onto $d(A)$.*

Proof. Let $m, n \in d(A)$. There exist $\mu, \nu \in A$ such that $m = d(\mu)$ and $n = d(\nu)$. If A is directed, there exists $\lambda \in A$ such that $\mu, \nu \leq \lambda$. Then $m, n \leq d(\lambda)$. Therefore $d(A)$ is directed. Suppose that $m \leq n$ and that $n = d(\lambda)$ with $\lambda \in A$. We write $n = mp$. By unique factorization, we can write $\lambda = \mu\pi$, where $d(\mu) = m$ and $d(\pi) = p$. If A is hereditary, then $\mu \in A$ and $m \in d(A)$. Therefore, $d(A)$ is hereditary.

Suppose that A is directed. Let $\mu, \nu \in A$ such that $d(\mu) = d(\nu)$. Let λ be a c.u.b. of (μ, ν) . By unique factorization of λ , we have the equality $\mu = \nu$. \square

⁵Unfortunately, the statement of [25, Lemma H.2] omits the observation that (F1) and (F2) must hold for every subnet of the original net.

Lemma 6.7. *Let Λ be a P -graph.*

- (a) *If $A \subset \Lambda$ is hereditary (resp., directed) and $n \in P$, the subset*

$$A \cdot n = \{ \nu \in \Lambda : \text{there exists } \mu \in \Lambda^n \text{ such that } \mu\nu \in A \}$$

is hereditary (resp., directed). If (r, d) is proper and A is directed and closed, then $A \cdot n$ is directed and closed.

- (b) *Assume (r, d) is proper and that P is a subsemigroup of a group Q . If $B \subset \Lambda$ is directed, hereditary and closed and if $\mu \in r(B)\Lambda$, then*

$$\mu B = \bigcup_{\nu \in B} \{ \lambda \in \Lambda : \lambda \leq \mu\nu \}$$

is directed, hereditary and closed. Moreover, if $A \cdot n$ is non-empty, there is a unique $\mu \in \Lambda^n \cap A$ such that $A = \mu(A \cdot n)$.

Proof. (a) Suppose that A is hereditary. Let $b \leq (a \cdot n)$ where $a \in A$. Then, $a = \mu(a \cdot n)$ where $d(\mu) = n$. Moreover, $a \cdot n = b \cdot c$ for some $c \in P$. Thus, $a = \mu(b \cdot c) = (\mu b) \cdot c$, and $\mu b \leq a$. Since A is hereditary, $\mu b \in A$, hence $b \in A \cdot n$. We have shown that $A \cdot n$ is hereditary.

Suppose that A is directed. Consider $a \cdot n$ and $b \cdot n$ where $a, b \in A$. We want a c.u.b. for $a \cdot n, b \cdot n$ in $A \cdot n$. Let c be a c.u.b. for a, b in A . Then $c \cdot n$ is defined and is a c.u.b. for $a \cdot n$ and $b \cdot n$.

Suppose that A is directed and closed, that a_i belongs to A and that $a_i \cdot n$ converges to some $b \in \Lambda$. We write $a_i = \mu_i(a_i \cdot n)$, where $d(\mu_i) = n$. Since we also have $r(\mu_i) = r(A)$, by (r, d) -properness, there is a subnet (μ_j) converging to some μ such that $r(\mu) = r(a)$, $s(\mu) = r(b)$ and $d(\mu) = n$. Then a_j converges to $a = \mu(b)$. Since A is closed, a belongs to A and $b = a \cdot n$ belongs to $A \cdot n$.

(b) To see that μB is directed, suppose that $\lambda \in B$ is a c.u.b. for $\nu_1, \nu_2 \in B$. Then $\mu\lambda$ is a c.u.b. for $\mu\nu_1, \mu\nu_2$. The set μB is hereditary by construction. To see that it is closed, consider a net λ_α in μB converging to λ . We distinguish two cases: if $d(\lambda) \leq d(\mu)$, then $d(\lambda_\alpha) \leq d(\mu)$ for α large enough. This implies $\lambda_\alpha \leq \mu$, hence $\lambda \leq \mu$. If $d(\lambda) \leq d(\mu)$ does not hold, there is a subnet λ_β for which $d(\lambda_\beta) \leq d(\mu)$ does not hold. Then, we can write $\lambda_\beta = \mu\nu_\beta$ with $\nu_\beta \in B$. Since $d(\lambda_\beta) = d(\lambda)$ for β large enough, this fixes $d(\nu_\beta)$ (if $P \subset Q$). We also have $r(\nu_\beta) = s(\mu)$. By (r, d) -properness, there is a converging subnet ν_γ . Its limit ν belongs to B because B is closed. We have $\lambda = \mu\nu$, hence λ is in μB .

If $A \cdot n$ is non-empty, there is $\lambda \in A$ such that $d(\lambda) \geq n$. We write $\lambda = \mu\nu$ with $\mu \in \Lambda^n$. Since $\mu \leq \lambda$, μ belongs to A . Let λ' be another element of A such that $d(\lambda') \geq n$. We can write $\lambda' = \mu'\nu'$. The unique factorization of a c.u.b. of (λ, λ') gives $\mu = \mu'$. Let us compare A and $\mu(A \cdot n)$. If $\lambda \in A$, the existence of a c.u.b. for (λ, μ) shows that $\lambda \in \mu(A \cdot n)$. Conversely, suppose that $\lambda \leq \mu\nu$ for $\nu \in A \cdot n$. There exists $\mu' \in \Lambda^n$ such that $\mu'\nu \in A$. By the above, $\mu' = \mu$, therefore $\mu\nu \in A$, hence $\lambda \in A$. \square

Proposition 6.8. *Let Λ be a P -graph, where P is a quasi-lattice ordered subsemigroup of a group Q and Λ is (r, d) -proper.*

- (a) *the set $\tilde{\Omega}$ of all hereditary and directed closed subsets of Λ is a closed subset of the space $\mathcal{C}(\Lambda)$ of closed subsets of Λ equipped with the Fell topology.*
- (b) *for all $\lambda \in \Lambda$, $F(\lambda) := \{ \mu \in \Lambda : \mu \leq \lambda \}$ belongs to $\tilde{\Omega}$;*
- (c) *$F(\Lambda)$ is dense in $\tilde{\Omega}$;*
- (d) *if $n \leq d(\lambda)$, $F(\lambda) \cdot n = F(\lambda \cdot n)$; if $s(\mu) = r(\lambda)$, $\mu F(\lambda) = F(\mu\lambda)$;*

(e) the map $F : \Lambda \rightarrow \mathcal{C}(\Lambda)$ is injective and continuous.

Proof. (a) Suppose that A_α converges to A in $\mathcal{C}(\Lambda)$ and that the A_α 's are hereditary and directed. Let us show that A is hereditary. Let $\lambda \in A$ and $\mu \leq \lambda$. Suppose that $\mu \notin A$. Since A is closed, there is an open set U and a compact set K such that $\mu \in U \subset K \subset \Lambda \setminus A$. The set $U\Lambda = \{\alpha\beta : \alpha \in U \text{ and } \beta \in \Lambda\}$ is open (because multiplication is assumed to be open and continuous), and contains λ . Thus we have $A \cap K = \emptyset$ and $A \cap U\Lambda \neq \emptyset$. There is α_0 such that for $\alpha \geq \alpha_0$, $A_\alpha \cap K = \emptyset$ and $A_\alpha \cap U\Lambda \neq \emptyset$. If λ_α belongs to $A_\alpha \cap U\Lambda$, there exists $\mu_\alpha \in U$ such that $\mu_\alpha \leq \lambda_\alpha$. Since A_α is hereditary, μ_α belongs to A_α . This contradicts $A_\alpha \cap K = \emptyset$.

Let us show that A is directed. Let μ, ν be in A . There exist nets μ_α and ν_α in A_α converging respectively to μ and ν . Let λ_α be a l.u.b. of (μ_α, ν_α) belonging to A_α . (Such a l.u.b. exists because A_α is directed and hereditary and P is quasi-lattice ordered). Since P is quasi-lattice ordered and (r, d) is proper, Λ is compactly aligned. Hence the net λ_α has a convergent subnet. Let λ be its limit. Since A_α converges to A , λ is in A (by (F1)). Since $\mu_\alpha, \nu_\alpha \leq \lambda_\alpha$, Lemma 6.5 gives $\mu, \nu \leq \lambda$. This shows that A is directed.

(b) The set $F(\lambda)$ is obviously hereditary and directed. According to Lemma 6.5, it is closed.

(c) Let A be a hereditary and directed closed subset of Λ . It will suffice to see that the net $(F(\lambda))_{\lambda \in A}$ converges to A . Let K be a compact set in Λ and U_1, \dots, U_n such that $A \cap K = \emptyset$ and $A \cap U_i \neq \emptyset$. Pick $\lambda_i \in A \cap U_i$ and let $\underline{\lambda}$ be a c.u.b. of $\lambda_1, \dots, \lambda_n$ in A . If $\lambda \geq \underline{\lambda}$, $F(\lambda) \cap U_i \neq \emptyset$ because it contains λ_i . Since $F(\lambda)$ is contained in A , its intersection with K is empty.

(d) Straightforward.

(e) Since we assume that $P \cap P^{-1} = \{e\}$, the relation \leq is an order relation, hence the injectivity of F . Suppose that $\lambda_\alpha \rightarrow \lambda$. Let K be a compact subset of Λ and let U_1, \dots, U_n be open subsets of Λ such that $F(\lambda) \cap K = \emptyset$ and $F(\lambda) \cap U_i \neq \emptyset$. If there is no α_0 such that $F(\lambda_\alpha) \cap K = \emptyset$ for all $\alpha \geq \alpha_0$, there are subnets $\mu_\beta \leq \lambda_\beta$ with $\mu_\beta \rightarrow \mu$ in K . This is not possible since we have then $\mu \leq \lambda$. By assumption, λ belongs to the open set $U_1\Lambda \cap \dots \cap U_n\Lambda$, therefore there exists $\alpha_1 \geq \alpha_0$ such that for all $\alpha \geq \alpha_1$, $\lambda_\alpha \in U_1\Lambda \cap \dots \cap U_n\Lambda$. Hence we eventually have $F(\lambda_\alpha) \cap U_i \neq \emptyset$. \square

We shall see later that the map $F : \Lambda \rightarrow \mathcal{C}(\Lambda)$ is not necessarily a homeomorphism onto its image (Remark 6.21(c)).

Recall that at the beginning of the subsection we defined $\Omega = \tilde{\Omega} \setminus \{\emptyset\}$. We now define

$$\begin{aligned} \Omega * P &= \{ (A, n) \in \Omega \times P : A \cdot n \neq \emptyset \} \\ &= \{ (A, n) \in \Omega \times P : \text{there exists } \lambda \in A \text{ such that } d(\lambda) \geq n \}, \end{aligned}$$

and $T : \Omega * P \rightarrow \Omega$ sending (A, n) to $A \cdot n$. As before, we define

$$U(n) = \{ A \in \Omega : (A, n) \in \Omega * P \} \quad \text{and} \quad V(n) = \{ A \cdot n : (A, n) \in \Omega * P \}$$

and we denote by $T_n : U(n) \rightarrow V(n)$ the map sending A to $A \cdot n$.

Definition 6.9. We define a *path* in Λ as a non-empty hereditary closed subset of Λ . The space Ω is called the *path space* of Λ . The above map T is called the *shift* on the path space.

Theorem 6.10. *Let Λ be a P -graph, where P is quasi-lattice ordered and Λ is (r, d) -proper. Let (Ω, P, T) be as above. Then, T is a directed action of P on Ω by partial local homeomorphisms.*

Proof. First, we observe that $U(n)$ and $V(n)$ are open subsets. Indeed, we have

$$U(n) = \{A \in \Omega : A \cap \bigcup_{m \geq n} \Lambda^m \neq \emptyset\} \quad \text{and} \quad V(n) = r^{-1}(s(\Lambda^n)),$$

where $r : \Omega \rightarrow \Lambda^{(0)}$ is the range map defined in Lemma 6.4. Let us show that $T_n : U(n) \rightarrow V(n)$ is a local homeomorphism. To show that T_n is continuous, suppose that A_α converges to A . We need to show that $A_\alpha \cdot n$ converges to $A \cdot n$. It suffices to show that every subnet $(A_\beta \cdot n)$ satisfies (F1) and (F2). Suppose that $\nu_\beta \in A_\beta \cdot n$ converges to ν . Then there is $\mu_\beta \in \Lambda^n$ such that $\mu_\beta \nu_\beta \in A_\beta$. Then $r(\mu_\beta) = r(A_\beta)$ converges to $r(A)$. By (r, d) -properness, there is a subnet $\mu_\gamma \in \Lambda^n$ which converges to some $\mu \in \Lambda^n$. Then $\mu\nu$ belongs to A and ν belongs to $A \cdot n$. Condition (F2) is clear: let $\nu \in A \cdot n$. There exists $\mu \in \Lambda^n$ such that $\mu\nu \in A$. There is a subnet A_β and $\lambda_\beta \in A_\beta$ such that λ_β converges to $\mu\nu$. Then $\lambda_\beta \cdot n \in A_\beta \cdot n$ converges to ν .

Let U be an open subset of Λ^n such that $s|_U : U \rightarrow s(U)$ is a homeomorphism; we denote by σ the inverse of $s|_U$. Then $\tilde{U} = \{A \in U(n) : A \cap U \neq \emptyset\}$ is open. Each $A \in \tilde{U}$ contains a unique $\mu \in U$ which is given by $\mu = \sigma(r(A \cdot n))$. Thus, the restriction of T_n to \tilde{U} is a bijection of \tilde{U} onto $T_n(\tilde{U}) = r^{-1}(s(U))$ having as inverse map $B \mapsto \sigma \circ r(B)B$. To show that this inverse map is continuous, it suffices to show that the product map $\Lambda^n * \Omega \rightarrow \Omega$ sending (μ, B) to μB is continuous. Consider a net (μ_α, B_α) converging to (μ, B) . We will show that $\mu_\alpha B_\alpha$ converges to μB by checking that every subnet $\mu_\beta B_\beta$ satisfies (F1) and (F2). For (F1), we proceed as in the proof of Lemma 6.7(b). Consider a net $\lambda_\beta \in \mu_\beta B_\beta$ converging to λ . We distinguish two cases: if $d(\lambda) \leq d(\mu)$, then $d(\lambda_\beta) \leq d(\mu_\beta)$ for β large enough. This implies $\lambda_\beta \leq \mu_\beta$, hence $\lambda \leq \mu$. If $d(\lambda) \leq d(\mu)$ does not hold, there is a subnet λ_γ for which $d(\lambda_\gamma) \leq d(\mu_\gamma)$ does not hold. Then, we can write $\lambda_\gamma = \mu_\gamma \nu_\gamma$ with $\nu_\gamma \in B_\gamma$. Since $d(\lambda_\gamma) = d(\lambda)$ and $d(\mu_\gamma) = d(\mu)$ for γ large enough, this fixes $d(\nu_\gamma)$. We also have $r(\nu_\gamma) = s(\mu_\gamma)$. By (r, d) -properness, there is a converging subnet ν_δ . Its limit ν belongs to B because B is closed. We have $\lambda = \mu\nu$, hence λ is in μB . Let us check (F2). Suppose that λ belongs to μB . Suppose first that $d(\lambda) \leq d(\mu)$. We have $n = pq$ with $p = d(\lambda)$. For β large enough, $d(\mu_\beta) = n$ and we have a unique factorization $\mu_\beta = \lambda_\beta \rho_\beta$ where $d(\lambda_\beta) = p$. By (r, d) -properness, λ_β has a subnet converging to some λ' ; since $\rho_\beta = \mu_\beta \cdot p$ converges to $\mu \cdot p$, we have $\mu = \lambda'(\mu \cdot p)$; by unique factorization, $\lambda' = \lambda$. Therefore λ_β converges to λ . Since λ_β belongs to $\mu_\beta B_\beta$, we are done if $d(\lambda) \leq d(\mu)$. Suppose now that $\lambda = \mu\nu$ where $\nu \in B$. There exists a subnet B_γ and $\nu_\gamma \in B_\gamma$ converging to ν . Then $\mu_\gamma \nu_\gamma$ belongs to $\mu_\gamma B_\gamma$ and converges to $\mu\nu$; therefore, $(\mu, B) \mapsto \mu B$ is continuous.

To see that the action is directed, consider

$$U(n) = \{A \in \Omega : \text{there exists } \lambda \in A \text{ such that } d(\lambda) \geq n\}.$$

Assume that $U(m) \cap U(n) \neq \emptyset$ and let $A \in U(m) \cap U(n)$. There exist $\mu, \nu \in A$ such that $d(\mu) \geq m$ and $d(\nu) \geq n$. Since A is directed, there exists $\lambda \in A$ greater than μ and ν . Then $d(\lambda)$ is greater than m and n . \square

Thus, under our assumptions on Λ and P , we can construct the semi-direct product groupoid $G(\Omega, P, T)$ according to Proposition 5.12.

Definition 6.11. Let Λ be a P -graph, where P is quasi-lattice ordered and Λ is (r, d) -proper and let T be the action of P on the path space Ω . The groupoid $G(\Omega, P, T)$ is called the *Toeplitz groupoid* of the topological higher rank graph Λ . Its C^* -algebra is called the *Toeplitz algebra* of Λ and denoted by $C^*(\Lambda)$.

This construction is the same as in the work of T. Yeend, for example [26]. The main differences are that we consider an arbitrary quasi-lattice ordered semigroup P rather than \mathbf{N}^d and that we make an explicit (rather than implicit) use of the Fell topology on a space of closed subsets to define the topological path space Ω . In [26], the path space Ω is denoted by X_Λ and the groupoid $G(\Omega, P, T)$, called the path groupoid, is denoted by G_Λ .

6.2. The boundary path space $\partial\Omega$. We continue to assume that P is quasi-lattice ordered and that Λ is (r, d) -proper. In particular, Λ is compactly aligned. The Cuntz-Krieger algebra of the P -graph Λ is described in [26] as the C^* -algebra of the reduction of $G(\Omega, P, T)$ to a closed invariant subset $\partial\Omega$ called the boundary path space. Let us describe the boundary path space in our presentation. Recall that the elements of Ω are the non-empty closed hereditary and directed subsets of Λ .

Definition 6.12. Let Λ be a P -graph. We say that $E \subset \Lambda$ is *exhaustive* if for all $\lambda \in \Lambda$ such that $r(\lambda) \in r(E)$, there exists $\mu \in E$ such that (λ, μ) has a c.u.b.

Definition 6.13. Let Λ be a P -graph.

- (a) Given $A \in \Omega$, we say that $\lambda \in A$ is *extendable in A* if for all $E \subset \Lambda$ which are exhaustive, compact and such that $r(E)$ is a neighborhood of $s(\lambda)$, there exists $\mu \in E$ such that $\lambda\mu \in A$.
- (b) We say that $A \in \Omega$ is a *boundary path* if all its elements are extendable in A .

We define the *boundary path space* $\partial\Omega$ as the subspace of all boundary paths.

Example 6.14 (Singly Generated Systems). Recall that this means a local homeomorphism $T : U \rightarrow V$, where U, V are open subsets of a locally compact Hausdorff space X and that

$$\Lambda = X * \mathbf{N} = \{ (x, n) \in X \times \mathbf{N} : x \in U(n) := \text{dom}(T^n) \}$$

The non-empty closed hereditary directed subsets of Λ are:

$$F(x, n) = \{ (x, m) : m \leq n \}$$

where $n \in \overline{\mathbf{N}} := \mathbf{N} \cup \{\infty\}$, $x \in U(n)$ if n is finite and $x \in U(\infty) = \bigcap U(n)$ if $n = \infty$. The boundary paths are:

$$F(x, \tau(x)) \quad \text{where} \quad \tau(x) = \sup\{n \in \mathbf{N} : x \in U(n)\}$$

Note that here the boundary paths are exactly the maximal paths.

Lemma 6.15. *Let $A \in \Omega$. If $\lambda' \in A$ is extendable in A , then every $\lambda \leq \lambda'$ is extendable in A .*

Proof. Let $E \subset \Lambda$ be exhaustive, compact and such that $r(E)$ is a neighborhood of $s(\lambda)$. We write $\lambda' = \lambda\xi$ and we let $n = d(\xi)$. We choose U open relatively compact neighborhood of ξ contained in Λ^n and such that $s|_{\overline{U}}$ is injective and $r(\overline{U}) \subset s(E)$. We define E' to be the set of elements μ' of Λ of minimal degree for which there

exist $\mu \in E$ and $\eta \in \overline{U}$ such that $\mu \leq \eta\mu'$. One checks that E' is closed. Since $r(\mu) \in r(\overline{U})$ and $d(\mu)$ lies in a finite set by the compact alignment property, E' is compact. We are going to show that E' is exhaustive and that $r(E') = s(\overline{U})$. By construction, $r(E') \subset s(\overline{U})$. Consider $\nu \in \Lambda$ such that $r(\nu) \in s(\overline{U})$. Pick $\eta \in \overline{U}$ such that $s(\eta) = r(\nu)$. Then $r(\eta\nu) = r(\eta) \in r(\overline{U}) \subset s(E)$. Since E is exhaustive, there exists $\mu \in E$ such that $(\eta\nu, \mu)$ has a c.u.b. This means the existence of $\alpha, \beta \in \Lambda$ such that $\mu\beta = (\eta\nu)\alpha$. We have then $\mu \leq \eta(\nu\alpha)$, hence the existence of $\mu' \in E'$ such that $\mu' \leq \nu\alpha$. In particular, $r(\nu) \in r(E')$, which shows that $r(E') = s(\overline{U})$. We have found $\mu' \in E'$ such that (μ', ν) has a c.u.b. This shows that E' is exhaustive. Since λ' is extendable in A , there exists $\mu' \in E'$ such that $\lambda'\mu' \in A$. By definition of E' , there is $(\mu, \eta) \in E \times \overline{U}$ such that $\mu \leq \eta\mu'$. Since η and ξ both belong to \overline{U} and have same source, $\eta = \xi$. Since $\lambda\mu \leq \lambda\xi\mu' = \lambda'\mu'$, $\lambda\mu \in A$. This shows that λ is extendable in A . \square

Proposition 6.16. *Let $\partial\Omega$ be the boundary path space of a P -graph Λ , where P is a quasi-lattice ordered subsemigroup of a group Q and Λ is (r, d) -proper. Then*

- (a) $\partial\Omega$ is a closed subset of Ω .
- (b) If $A \in \partial\Omega$ and $n \in P$ such that $A \cdot n \neq \emptyset$, then $A \cdot n \in \partial\Omega$.
- (c) If $B \in \partial\Omega$ and $\rho \in r(B)\Lambda$, then $\rho B \in \partial\Omega$.

Proof. (a) Suppose that $A_\alpha \rightarrow A$ and $A_\alpha \in \partial\Omega$. If A is not a boundary path, there exists $\lambda \in A$ and $E \subset \Lambda$ exhaustive, compact with $r(E)$ a neighborhood of $s(\lambda)$ such that $\lambda\mu \notin A$ for all $\mu \in E$. Let V be an open neighborhood of $s(\lambda)$ contained in $r(E)$. Let U be an open relatively compact neighborhood of λ such that $s(U) \subset V$ and $U \subset \Lambda^n$, where $n = d(\lambda)$. We have $A \cap U \neq \emptyset$ and $A \cap \overline{U}E = \emptyset$. Let us check this second assertion: if $\lambda'\mu \in A$, with $\lambda' \in \overline{U}$ and $\mu \in E$, then $\lambda' \in A$. Since $d(\lambda') = d(\lambda)$, $\lambda = \lambda'$. This is a contradiction. There exists α_0 such that for all $\alpha \geq \alpha_0$, $A_\alpha \cap U \neq \emptyset$ and $A_\alpha \cap \overline{U}E = \emptyset$. Let $\lambda_\alpha \in A_\alpha \cap U$. Then $r(E)$ is a neighborhood of $s(\lambda_\alpha)$ and for all $\mu \in E$ such that $r(\mu) = s(\lambda_\alpha)$, $\lambda_\alpha\mu$ does not belong to A . This contradicts the fact that A_α is a boundary path.

(b) Let A be a boundary path and $n \in P$ such that $A \cdot n$ is non-empty. Let us show that $A \cdot n$ is a boundary path. Recall that $\nu \in A \cdot n$ if and only if there exists $\rho \in \Lambda^n$ such that $\rho\nu \in A$. Let $\nu \in A \cdot n$ and $E \subset \Lambda$ exhaustive, compact with $r(E)$ a neighborhood of $s(\nu)$. There exists $\rho \in \Lambda^n$ such that $\lambda = \rho\nu \in A$. Since A is a boundary path, there exists $\mu \in E$ such that $\lambda\mu \in A$. Therefore $\nu\mu \in A \cdot n$.

(c) We first show that every $\lambda \in \rho B$ of the form $\lambda = \rho\nu$, where $\nu \in B$, satisfies the property. Indeed, let E be a subset of Λ which is exhaustive, compact and such that $r(E)$ is a neighborhood of $s(\lambda)$. Since $s(\lambda) = s(\nu)$ and B is a boundary path, there exists $\mu \in E$ such that $\nu\mu \in B$. Then $\lambda\mu = \rho(\nu\mu)$ belongs to ρB . We apply Lemma 6.15 to conclude that $\rho B \in \partial\Omega$. \square

Corollary 6.17. *Let Λ be a P -graph, where P is quasi-lattice ordered and Λ is (r, d) -proper. Let (Ω, P, T) and $G(\Omega, P, T)$ be as above. Then the boundary path space $\partial\Omega$ is a closed invariant subspace of Ω with respect to $G(\Omega, P, T)$.*

Proof. The equivalence relation on Ω induced by $G(\Omega, P, T)$ is precisely $A \sim B$ if and only if there exist $m, n \in P$ such that $A \cdot m$ and $B \cdot n$ are non-empty and equal. If A is a boundary path, so is $A \cdot m = B \cdot n$ by Proposition 6.16(b). Since $B = \rho(B \cdot n)$ for some ρ , B is a boundary path by part (c) of the same proposition.

Therefore, the space of boundary paths is invariant. We have shown above that it is closed. \square

Definition 6.18. The reduction $G(\partial\Omega, P, T)$ of the Toeplitz groupoid $G(\Omega, P, T)$ is called the *Cuntz-Krieger groupoid* of the topological higher rank graph Λ . Its C^* -algebra is called the *Cuntz-Krieger algebra* of Λ and denoted by $C^*(\partial\Lambda)$.

Note that $G(\partial\Omega, P, T)$ is the semi-direct product groupoid of the semigroup action $(\partial\Omega, P, T)$. Since the action on Ω is directed, so is the action on $\partial\Omega$. Hence Theorem 5.13 (and [1, Corollary 6.2.14]) give us the following.

Corollary 6.19. *Let P a quasi-lattice ordered subsemigroup of a group Q and Λ be a P -graph which is (r, d) -proper. If the group Q is amenable, then the Toeplitz groupoid $G(\Omega, P, T)$ and the Cuntz-Krieger groupoid $G(\partial\Omega, P, T)$ are amenable. Therefore the Toeplitz algebra $C^*(\Lambda)$ and the Cuntz-Krieger algebra $C^*(\partial\Lambda)$ are nuclear.*

6.3. Topological higher rank graphs coming from semigroup actions. We have seen that a semigroup action (X, P, T) gives the topological higher rank graph $\Lambda = X * P$. If P is quasi-lattice ordered, we can construct the groupoids $G(\Omega, P, T)$ and $G(\partial\Omega, P, T)$. (Recall that the graph of an action is always (r, d) -proper.) On the other hand, if the action is directed, we can construct the groupoid $G(X, P, T)$. It is then natural to compare these groupoids when the action is directed and P is quasi-lattice ordered. An important case is when P is both directed and quasi-lattice ordered, which means that P is lattice ordered.

Proposition 6.20. *Let (X, P, T) be a directed semigroup action, where P is a quasi-lattice ordered subsemigroup of a group Q . Then there is a P -equivariant homeomorphism of X onto $\partial\Omega$ which implements a groupoid isomorphism of $G(X, P, T)$ and $G(\partial\Omega, P, T)$.*

Proof. Given $x \in X$, $A(x) = x\Lambda = \{(x, n) \in X \times P : x \in U(n)\}$ is a closed subset of Λ . It is hereditary. It is directed because the action is directed. This defines a map $J : X \rightarrow \Omega$ sending x to $J(x) = x\Lambda$ which is injective. The map J is continuous. Indeed, let $K = L * F$, where L is a compact subset of X and F a finite subset of P and $U_i = V_i * \{p_i\}$, $i = 1, \dots, n$, where V_i is an open subset of X and $p_i \in P$. Then, $J(x) \cap K = \emptyset$ and $J(x) \cap U_i \neq \emptyset$ for all $i = 1, \dots, n$ if and only if

$$x \in (L^c \cup r(d^{-1}(F^c)) \cap V_1 \cap \dots \cap V_n,$$

which is open. The inverse map is continuous since it is the restriction to $J(X)$ of the range map $r : \Omega \rightarrow X$.

We claim that $J(x)$ is a boundary path. Because the action is directed, every subset E of Λ is exhaustive. Let us show that (x, m) , where $x \in U(m)$, is extendable in $J(x)$. Let E be a subset of Λ such that $x \cdot m = s(x, m) \in r(E)$. There exists $n \in P$ such that $(x \cdot m, n) \in E$. Then $x \in U(mn)$ and $(x, mn) = (x, m)(x \cdot m, n)$.

We also claim that every boundary path A is of the form $J(x)$, where $x = r(A)$. We have $A \subset J(x)$ by definition. Conversely, let $(x, n) \in J(x)$. Let $E = L * \{n\}$ where L is a compact neighborhood of x . Since A is extendable in A at $\lambda = (x, e) \in A$, there exists $\mu \in E$ such that $\lambda\mu \in A$. Necessarily $\mu = (x, n)$ and $(x, n) = (x, e)(x, n)$ belongs to A .

Let us show that the map $J : X \rightarrow \Omega$ is P -equivariant: for all $(x, m) \in X * P$, $(J(x), m) \in \Omega * P$ and $J(x \cdot m) = J(x) \cdot m$. Let $(x, m) \in X * P$. Since

$(x, m) = (x, m)(x \cdot m, e)$ belongs to $J(x)$, $J(x) \cdot m$ is non-empty. In fact, we have $J(x) \cdot m = J(x \cdot m)$: given that $x \in U(m)$, $(y, n) \in J(x) \cdot m$ means exactly that $y = xm$ and $x \in U(mn)$ while $(y, n) \in J(xm)$ means that $y = xm$ and $xm \in U(n)$.

Thus, the semigroup actions (X, P, T) and $(\partial\Omega, P, T)$ are isomorphic. One deduces that the semi-direct product groupoids $G(X, P, T)$ and $G(\partial\Omega, P, T)$ are isomorphic. \square

Remark 6.21. (a) Consider a locally compact semigroup action (X, P, T) . We are able to construct the semi-direct product groupoid $G(X, P, T)$, hence the C^* -algebra $C^*(G(X, P, T))$, when the action is directed. If the action is not directed but P is quasi-lattice ordered, we can introduce the topological higher rank graph $\Lambda = X * P$ and consider instead the semigroup action $(\partial\Omega, P, T)$, the groupoid $G(\partial\Omega, P, T)$ and the Cuntz-Krieger C^* -algebra $C^*(\partial\Lambda)$. Both constructions agree when they are possible.

(b) It is instructive to specialize the situation (a) to the case where X is reduced to a point. It turns out that this leads us to Wiener-Hopf (also called Toeplitz) C^* -algebras of semigroups. We have seen earlier that, in this case, the semidirect product $G(X, P, T)$ is PP^{-1} . If P is an Ore semigroup, then PP^{-1} is a group and the corresponding C^* -algebra is its group C^* -algebra. If P is not an Ore semigroup, we cannot even define a C^* -algebra. However, if P is quasi-lattice ordered, we can perform the construction of (a) which introduces the higher rank graph $\Lambda = P$. Nica shows in [14] that the corresponding path space Ω is the spectrum of a canonical diagonal C^* -subalgebra of the Wiener-Hopf algebra $\mathcal{W}(Q, P)$. In fact, he defines in Section 9 of the preprint version of [14] an étale groupoid \mathcal{G} , which he calls the Wiener-Hopf groupoid and which is exactly (up to an obvious isomorphism) our groupoid $G(\Omega, P, T)$. This Wiener-Hopf groupoid is only briefly mentioned in the subsection 1.5 of the published version [14]. The reduced C^* -algebra of this groupoid is the Wiener-Hopf C^* -algebra $\mathcal{W}(Q, P)$ while the non-degenerate representations of its full C^* -algebra are exactly the Nica-covariant representations of (Q, P) . He also shows that, when $P \subset Q$ is the free semigroup $SF_n \subset F_n$, the reduction to the boundary of the Wiener-Hopf groupoid, which we have denoted earlier by $G(\partial\Omega, P, T)$, is the Cuntz groupoid \mathcal{O}_n of [17]. Therefore its reduced and its full C^* -algebras coincide and are isomorphic to the Cuntz algebra O_n . One can also note that, in this particular case (b) of the general theory of higher rank graphs, the boundary path space $\partial\Omega$ is also studied by Crisp and Laca [5]. Their definition agrees with ours. We refer the reader to the work on semigroup C^* -algebras by X. Li [11, 12] and by Sundar [23] for recent developments.

(c) Let us justify the assertion made earlier that the map $F : \Lambda \rightarrow \mathcal{C}(\Lambda)$ is not necessarily a homeomorphism onto its image. Let (X, T) be a singly generated dynamical system in the sense of [18]: X is a locally compact space and T is a local homeomorphism from an open subset $\text{dom}(T)$ of X onto another open subset $\text{ran}(T)$ of X . For $n \in \mathbb{N}$, let $U(n) = \text{dom}(T^n)$. We view T as an action of \mathbb{N} on X , with $X * \mathbb{N} = \{(x, n) \in X \times \mathbb{N} : x \in U(n)\}$ and $xn = T^n x$. The associated \mathbb{N} -graph is $\Lambda = X * \mathbb{N}$. Let x_α be a net converging to x in X . Assume that there exists n such that $x_\alpha \in U(n)$ for all α and $x \in U(n-1) \setminus U(n)$. Then $F(x_\alpha, n)$ converges to $F(x, n-1)$ in $\mathcal{C}(\Lambda)$ but (x_α, n) does not converge to $(x, n-1)$ in Λ .

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